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Disjunctive Kriging Revisited: Part II[1]

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In order to find distributions other than infinitely divisible distributions which are suitable for disjunctive kriging, infinitesimal generators are used. In addition to distributions developed in Part I, this leads to development of suitable models for the beta (β), hypergeometric, and binomial distributions.

KEY WORDS: Nonlinear geostatistics, disjunctive kriging, beta distribution, hypergeometric distribution, binomial distribution, isofactorial.

INTRODUCTION

In the decade since the first paper on non-linear geostatistics appeared (Matheron, 1976), geostatisticians have had time to test the method and find its strengths and weaknesses. One problem to date has been that, in its present form, disjunctive kriging always has been associated with a transformation to a normal distribution, which is unsuited for use with data like uranium, which has a large peak of zero values, with discrete variables, or with grouped data as is found in size or density distributions. So a real need exists for new types of disjunctive kriging, particularly for "discrete disjunctive kriging."

The first note on disjunctive kriging (Matherson, 1973) gives the theory behind the method and shows how it can be used for data having one of the following distributions: normal distribution, gamma (γ), Poisson, or negative binomial. More importantly, general conditions for finding distributions suitable for disjunctive kriging are presented. These are that the joint distribution $f(x, y)$ can be expressed in an isofactorial form; that is

$$f(x, y) = \sum_{n=0}^{\infty} T_n \chi_n(x) \chi_n(y) g(x) g(y)$$

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where $g(\cdot)$ is the marginal distribution, T_n are constants, and $\chi_n(\cdot)$ are factors which, for simplicity, must be polynomials. A recent translation of this work is presented in a previous paper (Armstrong and Matheron, 1986).

One unfortunate limitation of this method for finding distributions suitable for disjunctive kriging was that it could be applied only to infinitely divisible distributions. This limitation can be overcome by using a different approach (infinitesimal generators). This was developed first for continuous distributions (Matheron, 1975a) and afterward for discrete distributions (Matheron, 1975b). The objective of this paper is to present an updated translation of those parts of these two research notes which are relevant directly to alternative types of disjunctive kriging.

THE INFINITESIMAL GENERATOR METHOD APPLIED TO CONTINUOUS DISTRIBUTIONS

In Part I, four distributions (normal, γ, Poisson, and negative binomial) were shown to have required isofactorial properties together with polynomial factors. Unfortunately, the method used for finding suitable models could be used only with infinitely divisible distributions.

Here, infinitesimal generators and the theory of semigroups are used to find other suitable models. In this section, the method is applied to continuous distributions; the discrete case is treated in the following section. In both cases, we give only an outline of the proof. Readers who wish to fill out the proof may find it helpful to consult a text on functional analysis (such as, Brezis, 1983), for the Hille–Yosida theorem in particular.

Let $g(x)$ be the marginal distribution. Working in the space $L^2(\mathbb{R}, g)$, we want to find a function $a(x)$ so as to write the differential operator Af associated with the stochastic process in the form

$$Af = af'' + \frac{f'}{g}\frac{d}{dx}(ag)$$

where $f' = \partial f/\partial x$.

Given this function $a(x)$, we can show that operator A is a negative Hermitian operator (in the sense that the scalar product $\langle Af, f\rangle = -\langle \sqrt{a}f', \sqrt{a}f'\rangle \leqslant 0$. (See Appendix A.) The evolution equation

$$\frac{\partial f_t}{\partial t} = Af = \frac{1}{g}\frac{\partial}{\partial x}\left[ag\frac{\partial f_t}{\partial t}\right] \tag{1}$$

is a type of heat equation.

Now, provided that A is closed and dense in $L^2(\mathbb{R}, g)$, a semigroup $P_t = e^{At}$ with A as its infinitesimal operator exists, and because we are dealing with a heat equation, this is a diffusion semigroup. Because $\int gAf_t = 0$, the density g is the ergodic limit of $P_t(x; dy)$ as $t \to \infty$.

The next step is to see whether the eigenfunctions associated with operator A include a series χ_n which forms an orthonormal basis for $L^2(\mathbb{R}, g)$. If this is the case

$$P_t\chi_n = e^{-\lambda_n t}\chi_n$$

where λ_n is the eigenvalue corresponding to χ_n.

Consequently for any function $f \in L^2(\mathbb{R}, g)$

$$P_t f = \sum e^{-\lambda_n t}\langle f, \chi_n \rangle \chi_n$$

The bivariate distribution $F_t(dx, dy) = g(x)\, P_t(dx, dy)\, dx$ can be written, therefore, in an isofactorial form

$$F_t(dx, dy) = \sum e^{-\lambda_n t}\chi_n(x)\, \chi_n(y)\, g(x)\, g(y)\, dx\, dy.$$

In other words, the Markov process associated with this semigroup is an isofactorial model.

A few examples should help clarify this approach.

Gaussian Process on the Whole Line

This corresponds to the case $a(x) = 1$ and $g(x) = (1/\sqrt{a\pi})\, e^{-x^2/2}$. Here

$$Af = f'' - xf'$$

which clearly is equal to $(1/g)(d/dx)(gf')$. The orthogonal polynomials relative to the normal distribution are Hermite polynomials $H_n(x)$. Moreover, because $AH_n = -nH_n$, eigenvalues are $\lambda_n = n$ and so the bivariate distribution $F_t(dx, dy)$ is $\Sigma\, \rho^n H_n(x)\, H_n(y)\, g(x)\, g(y)\, dx\, dy$ where the correlation coefficient ρ equals e^{-t}.

γ Process on R^+

This time $a(x) = x$ and $g(x) = [1/\Gamma(\alpha)]\, x^{\alpha-1}\, e^{-x}$. Consequently,

$$Af = xf'' + (\alpha - x)f'$$

The Laguerre polynomials

$$L(-n, \alpha, x) = 1 + \sum_{k=1}^{n} \frac{(-1)^k\, n(n-1)\cdots(n-k+1)x^k}{\alpha(\alpha+1)\cdots(\alpha+k-1)\, k!}$$

are eigenfunctions associated with eigenvalue $-\lambda_n = -n$. Letting l_n denote the normed polynomial, the representation for the bivariate distribution is

$$F_t = \frac{\sum e^{-nt} l_n(x)\, l_n(y)\, x^{\alpha-1}\, y^{\alpha-1}\, e^{-x-y}}{\Gamma(\alpha)\,\Gamma(\alpha)}$$

—NOTE—

The definition given here for Laguerre polynomials is not the same as in earlier notes. They differ by a multiplicative factor $(-1)^n(\alpha + n - 1)(\alpha + n - 2) \cdots \alpha$. The new definition is used because it simplifies considerably the form of the recurrence relation used to obtain Laguerre polynomials from the preceding two polynomials.

β Process on (0, 1)

In this case, $a(x) = x(1 - x)$ and

$$g(x) = \frac{\Gamma(\alpha + \beta)\, x^{\alpha - 1}(1 - x)^{\beta - 1}}{\Gamma(\alpha)\, \Gamma(\beta)} \qquad 0 \leq x \leq 1$$

The expression for the operator A is

$$Af = x(1 - x)f'' + [\alpha - (\alpha + \beta)\, x]f'$$

The orthogonal polynomials are Jacobi polynomials

$$F(n, \alpha + \beta + n - 1, \alpha; x)$$
$$= 1 + \sum_{k=1}^{n} \frac{(-1)^k\, n!\, (\alpha + \beta + n + k - 2)!\, (\alpha - 1)!\, x^k}{(n - k)!\, (\alpha + \beta + n - 2)!\, (\alpha + k - 1)!\, k!}$$

which are associated with eigenvalues

$$-\lambda_n = -n(n + \alpha + \beta - 1)$$

If χ_n denotes the normed polynomial

$$F_t(dx, dy) = \sum e^{-n(n + \alpha\beta - 1)} \chi_n(x)\, \chi_n(y)\, g(x)\, g(y)\, dx\, dy$$

The form of normed factors is rather curious

$$1 + \cdots + \frac{(-1)^n\, \Gamma(\alpha)\, \Gamma(\alpha + \beta + 2n - 1)}{\Gamma(\alpha + n)\, \Gamma(\alpha + \beta + n - 1)}\, x^n$$

The family of polynomials can be expressed in another way

$$x^{1-\alpha}(1 - x)^{1-\beta} \frac{d^n}{dx^n} [x^{\alpha + n - 1}(1 - x)^{\beta + n - 1}]$$

In this case, the constant needed to normalize polynomials is

$$\frac{\Gamma(\alpha + \beta)\, \Gamma(\alpha + n)\, \Gamma(\beta + n)}{\Gamma(\alpha)\, \Gamma(\beta)\, \Gamma(\alpha + \beta + 2n - 1)}$$

One more important difference between this distribution and the preceding two is that we no longer have

$$T_n = \rho^n = e^{-nt}$$

Instead

$$T_n = e^{-n(\alpha+\beta+n-1)t}$$

INFINITESIMAL GENERATOR METHOD APPLIED TO DISCRETE DISTRIBUTIONS

In the preceding section, isofactorial models with polynomial factors were found for continuous distributions from Markov processes where the infinitesimal generator was local. In this section, similar models are obtained for discrete distributions. The local nature of infinitesimal generator A is replaced by a condition limiting possible transitions to adjoining states, that is $i \to i + 1$ or $i \to i - 1$. Consequently

$$(Af)i = -(a_i + b_i)f_i + a_i f_{i+1} + b_i f_{i-1}$$

where a_i is the probability of transition $i \to i + 1$ and b_i is the probability of transition $i \to i - 1$.

Three different types of processes can be distinguished, depending on values of a_i and b_i.

1. If a_i and b_i are strictly positive for $i = 0, \pm 1, \pm 2, \ldots$, i can vary from $-\infty$ to $+\infty$. This case will not be treated here (reasons for this choice will become apparent in the next paragraph.)
2. If $b_0 = 0$ and $b_i > 0$ for $i = 1, 2, \ldots$, and $a_i > 0$ for all i, i varies from 0 to ∞ (infinite case).
3. If $b_0 = 0$ and $a_N = 0$, and if $b_i > 0$ for $i = 1, \ldots, N$ and $a_i > 0$ for $i = 0, \ldots, N - 1$, i varies from 0 to N (finite case).

In addition to this, the process must be ergodic, so a probability $W = \{w_i\}$ must exist such that

$$-(a_i + b_i)w_i + a_{i-1}w_{i-1} + b_{i+1}w_{i+1} = 0$$

On rewriting

$$b_{i+1}w_{i+1} - a_i w_i = b_i w_i - a_{i-1}w_{i-1}$$

and clearly

$$b_{i+1}w_{i+1} = a_i w_i \tag{2}$$

If a_i and b_i are strictly positive, eq. 2 is true provided that $a_i w_i$ and $b_i w_i$ tend to 0 as i tends to $-\infty$. However, in the two cases of interest to us (finite and infinite cases), we have $b_0 = 0$ and so eq. 2 is true. No additional hypotheses are required. Therefore, the limiting distribution (if it exists) is defined by

$$w_n = \frac{a_0}{b_1}\frac{a_1}{b_2} \cdots \frac{a_{n-1}}{b_n} w_0 \tag{3}$$

and the condition for its existence is that $\Sigma\, w_n < \infty$.

Condition (2) also means that the process is reversible, because $w_i P_{ij}(t) = w_j P_{ji}(t)$, where $P_{ij}(t)$ is the probability of going from i to j in time t.

Condition for Polynomial Factors

Because the process is reversible, polynomial factors exist if and only if

I. Polynomials belong to $L^2(\mathbb{R}, W)$; that is

$$E(i^n) = \sum w_i i^n < \infty$$

II. For each n, A_i^n is a polynomial of degree n in i (where $n = 0, 1, 2, \ldots, \infty$ or $n = 0, 1, \ldots, N$ as the case may be).

Taking condition II first

$$A_i^n = a_i[(i + 1)^n - i^n] + b_i[(i - 1)^n - i^n]$$

So for $n = 1$

$$A_i = a_i - b_i$$

and this must be a 1st degree polynomial in i. For $n = 2$

$$A_i^2 = a_i[2i + 1] + b_i[-2i + 1]$$

and this must be a 2nd degree polynomial in i.

These two conditions are satisfied if

$$a_i - b_i \text{ is linear} \quad \text{and}$$

$$a_i + b_i \text{ is quadratic}$$

that is, if a_i and b_i are of the form

$$\begin{aligned} a_i &= a_0 + \alpha i + \gamma^2 \\ b_i &= b_0 + \beta i + \gamma^2 \end{aligned} \tag{4}$$

Moreover, $b_0 = 0$ because we are not concerned with cases where i goes from $-\infty$ to ∞. If a_i and b_i are of this form, condition (2) is satisfied for all n.

Condition I is satisfied also in the infinite case. (Obviously it is for the finite one). In the case where a_i and b_i are both strictly positive, i.e., where i varies from $-\infty$ to $+\infty$, we find that $E[i^n] = \infty$ for sufficiently large n and consequently that no models with polynomial factors exist. This is the reason that this case has not been considered.

Linear Models: $\gamma = 0$

Here

$$a_i = a_0 + \alpha i$$

$$b_i = \beta i$$

Now a_0 must be strictly positive or else the process stops; similarly, so must β. Three cases must be considered: $\alpha > 0$, $\alpha = 0$, $\alpha < 0$ which lead, respectively, to the negative binomial, Poisson, and binomial distributions. (This progression is hardly surprising when limit relations between different distributions are remembered).

Case $\alpha > 0$: Negative Binomial Distribution

From eq. 3, clearly

$$W_n = \left(\frac{\alpha}{\beta}\right)^n \left[\frac{a_0}{\alpha}\left(\frac{a_0}{\alpha} + 1\right) \cdots \left(\frac{a_0}{\alpha} + n - 1\right)\right] \frac{W_0}{n!}$$

Putting $p = \alpha/\beta$, $\nu = a_0/\alpha$

$$\sum w_n s^n = (1 - ps)^{-\nu} W_0 \qquad s < \frac{1}{p}$$

Therefore, the limiting distribution W exists iff $p < 1$; that is, if $\alpha < \beta$. In that case, the generating function is

$$G(s) = \frac{q^\nu}{(1 - ps)^\nu}$$

where $q = 1 - p$. Because $G(s)$ can be differentiated as many times as is desired, the polynomials belong to $L^2(\mathbb{R}, W)$. Provided $\alpha < \beta$, an isofactorial model therefore exists with polynomial factors and with the negative binomial

$$W_n = q^\nu \frac{\Gamma(\nu + n)}{\Gamma(\nu)} \frac{p^n}{n!}$$

as its marginal distribution.

Polynomials have to be determined explicitly. To do this, we need eigenvalues of operator A that can be found by finding the coefficient of i^n in the expression for A_i^n

$$A_i^n = (a_0 + \alpha i)\,[(i + 1)^n - i^n] - \beta i[i^n - (i - 1)^n]$$
$$= n(\alpha - \beta)i^n + \text{polynomial of degree} < \text{n}.$$

Consequently

$$\lambda_n = -n(\beta - \alpha) = -n\beta q$$

So if $\chi_n(i)$ denotes the normed factor of degree n, the bivariate distribution $F_{ij}(t)$ is

$$F_{ij}(t) = W_i W_j \sum_{n=0}^{\infty} \rho^n \chi_n(i)\, \chi_n(j) \tag{5}$$

where $\rho = e^{-\beta q t}$.

Moreover,

$$F_{ij}(t) = W_i P_{ij}\,(t)$$

where $P_{ij}(t)$ is the translation matrix which is the resolution of the second Kolmogorov equation

$$\frac{d}{dt} P_{ij}(t) = -(a_j + b_j)\, P_{ij}(t) + a_{j-1} P_{ij-1}(t) + b_{j+1} P_{ij+1}(t) \tag{6}$$

Putting

$$b_i(s, t) = \sum_j P_{ij}(t) s^j$$

Equation 5 then becomes

$$(\partial G/\partial t) + (1 - s)(\alpha s - \beta)(\partial b/\partial s) = -a_0(1 - s)$$

with $G_i(s, \sigma) = s^i$. Integrating this gives

$$G_i(s, t) = \left[\frac{(1 - ps - \rho(1 - s)}{1 - ps - p\rho(1 - s)}\right]^i \left[\frac{q}{1 - ps - p\rho(1 - s)}\right]^\nu$$

Bivariate generating functions in terms of s and σ (t is implicit) can be deduced from eq. 6

$$G(s, \sigma) = \sum_{i,j} F_{ij} s^j \sigma^i$$
$$= \sum W_i \sigma^i G_i(s, t)$$
$$= \left[\frac{q^2}{(1 - ps)(1 - p\sigma) - \rho p(1 - s)(1 - \sigma)}\right]^\nu$$

This can be expanded in terms of ρ as

$$G(s, \sigma) = \left[\frac{q^2}{(1 - ps)(1 - p\sigma)}\right]^{\nu} \sum_{0}^{\infty} \frac{\Gamma(\nu + n)\, p^n \rho^p}{\Gamma(\nu)\, n!} \frac{(1 - s)^n (1 - \sigma)^n}{(1 - \rho s)^n (1 - \rho\sigma)^n}$$

From eq. 5 we also have

$$G(s, \sigma) = \sum_{n} \rho^n \sum_{i} W_i \chi_n(i) s^i \sum_{j} W_j \chi_n(j) \sigma^j$$

Because $\rho = e^{-\beta q t}$ varies from 0 to ∞, we can equate terms in these two expansions. This shows that the generating function of $\chi_n(i) W_i$ is

$$\sum \chi_n(i) W_i s^i = q^{\nu} \sqrt{\frac{\Gamma(\nu + n) p^n}{\Gamma(\nu)\, n!}} \frac{(1 - s)^n}{(1 - ps)^{n+\nu}}$$

Polynomial $\chi_n(i)$ can be deduced from this to be

$$\left[\frac{\Gamma(\nu + n)}{\Gamma(\nu)} \frac{p^n}{n!}\right]^{1/2} \sum_{k=0}^{n} (-1)^k \frac{{}^nC_k \Gamma(\nu + i + n - k)}{\Gamma(\nu + i)} \frac{i(i - 1) \cdots (i - k + 1)}{p^k}$$

Case $\alpha = 0$: Poisson Distribution

If $\alpha = \gamma = b_0 = 0$ in (4), then $a_i = a_0$ and $b_i = \beta i$ where $\beta > 0$. From (3), the limiting distribution still exists and is a Poisson distribution with parameter $\theta = (a_0/\beta)$. The corresponding stochastic process describes a queuing process with an infinite number of servers, when arrival times follow a Poisson distribution with parameter a_0 and the service time distribution is exponential with parameter β. The infinitesimal generator is

$$(Af)_i = a_0(f_{i+1} - f_i) - b_i(f_i - f_{i-1})$$

Consequently, eigenvalue λ_n associated with polynomial factor χ_n (of degree n) is $\lambda_n = -n\beta$.

So bivariate distribution $F_{ij}(t) = W_i P_{ij}(t)$ can be written as

$$F_{ij} = W_i W_j \sum_{n=0}^{\infty} \rho^n \chi_n(i)\, \chi_n(j) \tag{7}$$

where $\rho = e^{-\beta t}$.

Using Kolmogorov's second equation

$$\sum_{j} P_{ij}(t) s^j = (1 - \rho + p\rho)^j\, e^{-\theta(1-\rho)(1-s)}$$

and hence the generating function for the bivariate distribution is deduced

$$\begin{aligned} G(s, \sigma) &= \sum_{ij} F_{ij} s^j \sigma^j \\ &= \sum_{ij} W_i P_{ij} \sigma^i s^j \\ &= e^{-\theta(1-s) - \theta(1-\sigma) + \rho(1-s)(1-\sigma)} \end{aligned}$$

We now compare this with the expression for the generating function obtained from eq. 7

$$G(s, \sigma) = \sum_n \rho^n \sum_i W_i \chi_n(i) \sigma^i \sum_j W_j \chi_n(j) s^j$$

Identifying coefficients of terms in ρ^n gives

$$\sum W_i \chi_n(i) s^i = \sqrt{\frac{\theta^n}{n!}} (1 - s)^n e^{\theta(1-s)}$$

$$= (-1)^n \sqrt{\frac{\theta^n}{n!}} \frac{d^n}{d\theta^n} e^{\theta(1-s)}$$

Consequently

$$W_i \chi_n(i) = (-1)^n \sqrt{\frac{\theta^n}{n!}} \frac{d^n}{d\theta^n} \left(\frac{\theta^i}{i!} e^{-\theta}\right)$$

Therefore

$$\chi_n(i) = (-1)^n \sqrt{\frac{\theta^n}{n!}} \frac{i!\, e^\theta}{\theta^i} \frac{d^n}{d\theta^n} \left(\frac{\theta^i}{i!} e^{-\theta}\right)$$

Case $\alpha < 0$: Binomial Distribution

If α is negative, $a_i = a_0 + \alpha i$ would become negative for sufficiently large values of i. As $\{i\}$ therefore must be finite, a_i must be zero for some value of $i = N$, and the process is restrained to interval $(0, N)$. We therefore have

$$a_i = a(N - i) \qquad b_i = bi$$

From eq. 3, the limiting distribution is a binomial distribution with parameters p and N where

$$p = \frac{a}{a + b}$$

The infinitesimal generator is

$$(Af)_i = a(N - 1)\, [f_{i+1} - f_i] - bi[f_i - f_{i-1}]$$

and so the eigenvalue associated with χ_n is

$$\lambda_n = -n(a + b)$$

Consequently, the bivariate distribution is

$$F_{ij}(t) = W_i W_j \sum \rho^n \chi_n(i)\, \chi_n(j) \tag{8}$$

where $\rho = e^{-(a+b)c}$ and $W_i = \binom{N}{i} p^i (1 - p)^{N-i}$.

The expression for $\Sigma_{j=0}^{\infty} P_{ij}(t)s^j$ also can be obtained directly to be

$$[q(1 - \rho) + (p + \rho q)s]^i [q + \rho p + p(1 - \rho)s]^{N-i}$$

where $q = 1 - p$.
Consequently

$$G(s, \sigma) = \sum W_i P_{ij}(t) s^j \sigma^i$$
$$= [(q + ps)(q + p\sigma) + \rho pq(1 - s)(1 - \sigma)]^N$$

Expanding this in powers of ρ gives

$$G(s, \sigma) = (q + ps)^N (q + p\sigma)^N \sum_0^\infty \rho^n p^n q^n \frac{N}{n} \left[\frac{(1 - s)(1 - \sigma)}{(q + ps)(q + p\sigma)}\right]^n$$

Another expression for $G(s, \sigma)$ can be obtained directly from eq. 8. Identifying terms in the two expressions gives

$$\sum W_i \chi_n(i) s^i = \sqrt{\binom{N}{n} p^n q^n}\, (1 - s)^n (q + ps)^{N-n}$$
$$= \frac{n!}{N!} \sqrt{\binom{N}{n} p^n q^n}\, \frac{d^n}{dq^n} [q(1 - s) + s]^N$$

Consequently,

$$W_i \chi_n(i) = \frac{n!}{N!} \sqrt{\binom{N}{n} p^n q^n}\, \frac{d^n}{dq^n} \left[\binom{N}{i} (1 - q)^i\ q^{N-i}\right]$$

Hence

$$\chi_n(i) = \sqrt{\frac{n!\, p^n q^n}{N!\ (N - n)!}}\, \frac{1}{(1 - q)^i\ q^{N-i}} \frac{d^n}{dq^n} [q^{N-i}(1 - q)^i]$$

Case $\gamma \neq 0$: Hypergeometric Distribution

In this case

$$a_i = a_0 + \alpha i + \gamma_i^2$$
$$b_i = \beta i + \gamma_i^2$$

where $\gamma \neq 0$. In the infinite case $\gamma > 0$, $a_0 > 0$, and a_i, $b_i > 0$ for $i = 1, 2, \ldots$, a limiting distribution W may be obtained provided that certain conditions are satisfied, but even then polynomials of degree n need not belong to $L^2(R, W)$ when n becomes large. Consequently, no polynomial factors exist.

However, in the finite case, a solution exists. Suppose $0 \leq i \leq N$. Then, changing notation slightly, we have

$$a_i = (N - i)\,[a + \gamma(N - i)]$$

$$b_i = i[b + \gamma i]$$

where $a + \gamma > 0$, $a + N\gamma > 0$; $b + \gamma > 0$, $b + N\gamma > 0$.

From (3), the limiting distribution is of the form

$$W_n = \frac{N(N-1)\cdots(N-n+1)}{n!}\;\frac{\left(\frac{a}{\gamma} + N\right)\cdots\left(\frac{a}{\gamma} + N - n + 1\right)}{\left(1 + \frac{b}{\gamma}\right)\cdots\left(n + \frac{b}{\gamma}\right)}\,W_0$$

Putting $(a/\gamma) + N = \alpha$ and $(b/\gamma) = \beta$ gives

$$W_n = \frac{(-N)\cdots(-N+n-1)}{n!}\;\frac{(-\alpha)(-\alpha+1)\cdots(-\alpha+n-1)}{\beta(\beta+1)(\beta+n-1)}\,W_0$$

Because $\Sigma\, W_n = 1$

$$W_0 = \frac{\Gamma(\alpha + N)\,\Gamma(\beta + N)}{\Gamma(\beta)\,\Gamma(\alpha + \beta + N)}$$

Hence

$$W_n = \binom{N}{n}\frac{\Gamma(\beta + N)\,\Gamma(\alpha + N)}{\Gamma(\beta + n)\,\Gamma(\alpha + \beta + N)}\,\alpha(\alpha - 1)\cdots(\alpha - n + 1)$$

Polynomial factors still have to be found explicitly, but as this is much more complicated than in preceding cases, it will be left until it is needed for practical applications.

CONCLUSION

Isofactorial models presented in this paper and its predecessor permit disjunctive kriging when data are not normally distributed. Continuous distributions such as γ and β distributions may be useful for modeling long-tailed distributions, thus obviating need for an anamorphosis as used in traditional gaussian distinctive kriging. Discrete isofactorial models have a wide range of potential uses with discrete data or grouped data. Only experience will show how useful they really are in practice.

Before concluding, potential users should note two limitations in work presented in this paper. The first is of a numerical nature. Whereas isofactorial models with polynomial factors are good to have, these are of little practical interest unless polynomials can be calculated quickly and efficiently. Some sort of recurrence relation is needed therefore. These exist and can be found in any standard reference on orthogonal polynomials.

The second limitation concerns "change of support." One great advantage of the normal distribution is that it facilitated models of change of support, which are needed for point-block models and block-block models. In this work, nothing has been said about how these can be developed for isofactorial models. Interested readers can consult some recent work (Matheron, 1982, 1984, and 1985).

APPENDIX A

We want to show that operator A defined by

$$Af = af'' + \frac{f'}{g}\frac{d}{dx}(ag) = \frac{1}{g}\frac{d}{dx}(agf')$$

is a negative Hermitian operation; that is

$$\langle Af, f\rangle = -\langle \sqrt{a}f', \sqrt{a}f'\rangle \leqslant 0$$

Suppose that the marginal density $g(x)$ is sufficiently regular and is concentrated on the (possibly infinite) interval $[b, c]$. Let $a(x)$ be a strictly positive function on this interval and let $a(b) = a(c) = 0$. Moreover we suppose that product $a(x)\, g(x)$ is differentiable in this interval.

If f is twice differentiable on $[b, c]$ and if function φ is differentiable on the interval, then using integration by parts we can show that

$$\int_b^c \left\{ a(x)\, f''(x) + \frac{f'(x)}{g(x)}\frac{d}{dx}[a(x)\, g(x)] \right\} \varphi(x)\, g(x)\, dx$$

$$= -\int_b^c f'(x)\, \varphi'(x)\, a(x)\, g(x)\, dx$$

The result then follows.

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