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Mathematical Geology, Vol. 18, No. 8, 1986

Isofactorial Models for Granulodensimetric Data[1]

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Existing isofactorial models developed for disjunctive kriging using a cutoff grade on one variable are extended to the bivariate case which arises when dealing with granulo-densimetric data, such as are obtained from coal washing or mineral processing.

KEY WORDS: bivariate isofactorial models, disjunctive kriging, granulodensimetric data, coal washing, mineral processing.

NEED FOR ACCURATE PREDICTIONS

Widespread acceptance of kriging in the mining industry is due to its ability to provide accurate estimates of reserves. Linear geostatistics and kriging give reliable estimates of grades of blocks whereas nonlinear geostatistics (disjunctive kriging, multi-Gaussian kriging, etc.) can be used to estimate the percentage of selective mining units with a grade above a certain cutoff. These predictions are of great help to mine planners who have to maximize profits in an increasingly competitive world.

Although a good deal of theoretical research and practical testing has been done on these subjects, the problem of predicting recovery per block, after initial separation of saleable material from waste, generally has been neglected. Taking coal as an example, many sophisticated mathematical models and computer programs have been developed to aid mine planning (e.g., by simulating exploitation block by block) but these generally stop at entry to the wash plant. A different set of methods and models are used to predict results of the separation of the run-of mine material into coal and waste. However, the problem of predicting recovery and recovered ash and sulfur levels for blocks of coal in situ has attracted little attention. As the majority of coal mines have to treat their coal (by crushing and washing it) to reduce its ash and sulfur levels, pre-

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dictions of recovery after washing would be a tremendous aid in decision-making.

The importance of crushing and washing procedures can be seen from Table 1, which presents percentage recovery as a function of density of liquid used for separation for two size fractions.

(Note: gravity methods commonly used for separating "coal" from waste rely on the difference in density between the two. However, flotation methods used for separating fines depend on surface properties of the particles rather than their densities; these are outside the scope of this article).

Any attempt at predicting block recoveries and recovered quality must take into account both size and density. One unfortunate limitation of work to date on this subject (Armstrong, 1980, 1984; Kim, Barua, and Baafi, 1982 is that only one aspect was taken into account.

The aim of this paper is to present some simple bivariate models which may prove useful in coal washing and mineral processing contexts where recovery estimates depend on two factors. As disjunctive kriging can be used to predict recovered reserves when selection depends on one factor (grade of the block), it seems natural to try to develop a suitable isofactorial distributional model. Models presented here are only for discrete distributions. The reasons for not considering continuous distributions at the outset are first that reversibility conditions found for the discrete case are fairly complicated and would be worse in the continuous case, and second that data collection procedures for size and density distributions give rise to data which are grouped into a relatively small number of classes; this effectively hides the continuous nature of the data. We do envisage extending these results to the continuous case later.

Matheron (1973, 1975a,b, 1980, 1984, 1985a,b) has presented univariate isofactorial models related to several discrete distributions. Bivariate isofactorial models presented here are a natural extension of this work, particularly of Matheron (1975b and 1985a). However, as often happens when results for the univariate (i.e., one-dimensional) case are being extended to the bivariate case, new problems are encountered. Whereas conditions for having an ergodic dis-

Table 1. Percentage Coal Recovered as a Function of Liquid Density

	1-1.5 cm (%)	10 cm (%)
<1.3	20	17
<1.35	40	37
<1.4	48	51
<1.45	55	60
<1.5	60	65
<1.55	63	69
<1.6	65	71

tribution were fairly simple for the univariate case, they become much more complicated here and are replaced by an assumption of reversibility of the process. Second, conditions used in the univariate case to guarantee polynomial factors relied on existence of an ordering relation for the degree of polynomials x^n. The lack of an ordering relation in two dimensions means that no natural way of ordering polynomial terms $x^n y^m$ exists, and hence that the argument used for the univariate case cannot be transposed to the bivariate case.

We now go on to describe the discrete bivariate stochastic process which is the basis for our models.

DISCRETE BIVARIATE PROCESS

Consider a discrete bivariate process. Suppose that possible states for the first variable are denoted by i and by j for the second. Suppose, as is usual for Markov chains, that the only possible transitions are to neighboring states (i.e., $i \to i+1$, $i \to i-1$, $j \to j+1$, $j \to j-1$). So eight transitions each with its associated probability are possible

$$\begin{aligned}
a_{ij} &= \Pr\,(i \to i+1, j \to j) \\
b_{ij} &= \Pr\,(i \to i-1, j \to j) \\
c_{ij} &= \Pr\,(i \to i, j \to j+1) \\
d_{ij} &= \Pr\,(i \to i, j \to j-1) \\
e_{ij} &= \Pr\,(i \to i+1, j \to j+1) \\
f_{ij} &= \Pr\,(i \to i+1, j \to j-1) \\
g_{ij} &= \Pr\,(i \to i-1, j \to j-1) \\
h_{ij} &= \Pr\,(i \to i-1, j \to j+1)
\end{aligned} \tag{1}$$

HOW TO FIND ISOFACTORIAL MODELS

Three main steps in finding isofactorial models with polynomial factors are:

1. Working out implications for transition probabilities of assumptions of reversibility of the process.
2. Finding limiting distributions for processes with polynomial factors.
3. Finding an isofactorial representation of these distributions and, if possible, determining the recurrence relation between factors.

The approach follows the methodology used by Matheron (1975b, 1980) for univariate distributions. Readers who are not familiar with these papers may find it useful to consult them or their English translations (Armstrong and Matheron, 1986). Following this approach, we first find the infinitesimal generator A for the process

$$(A\varphi) = -(a_{ij} + b_{ij} + \cdots + h_{ij})\,\varphi_{i,j} + a_{ij}\,\varphi_{i+1,j} + b_{ij}\,\varphi_{i-1,j} + c_{ij}\,\varphi_{i,j+1} + \cdots + h_{ij}\,\varphi_{i-1,j-1} \tag{2}$$

As with the univariate case, several different types of processes arise. If all a_{ij} and b_{ij} (for a fixed value of j) are strictly positive, then for that j, i can vary from $-\infty$ to ∞. This case will not be treated here because it seems to be of little practical importance and, more importantly, because no models with polynomial factors in the univariate equivalent of this case existed. If $b_{0j} = 0$ and $b_{ij} > 0$ for $i = 1, 2, \ldots$ and if $a_{ij} > 0$ for $i = 0, 1, 2, \ldots$, i varies from 0 to $+\infty$ (infinite case). For this to be internally consistent, h_{0j} and g_{0j} also must be zero, or else a nonzero probability of i being negative exists. A similar argument applied to the other index shows that f_{i0} and g_{i0} must be zero also. If $b_{0j} = 0$ for all j (and also h_{0j} and g_{0j}) and if $a_{Nj} = 0$ (and also e_{Nj} and f_{Nj}), then i can take values $0, 1, \ldots, N$ (finite case).

As the same reasoning can be applied to the other index j, three cases would seem to be possible

1. $i \in \{0, 1, \ldots, \infty\}$
   $j \in \{0, 1, \ldots, \infty\}$
2. $i \in \{0, 1, \ldots, \infty\}$
   $j \in \{0, 1, \ldots, N\}$ or vice versa
3. $i \in \{0, 1, \ldots, N_1\}$
   $j \in \{0, 2, \ldots, N_2\}$

ASSUMPTION OF REVERSIBILITY

In the univariate case, Matheron developed a condition for the process to be ergodic which also guaranteed its reversibility. This equivalence between reversibility and the existence of a limiting distribution no longer holds in two dimensions. But, for simplicity, the process will be assumed to be reversible. Reasonably one may ask whether this assumption is acceptable for size distributions. As far as time-dependent processes (e.g., for grinding circuits or agglomeration of particles) are concerned, the answer almost certainly is no. However, if the process is location-dependent, the assumption seems reasonable because moving in space does not have the same connotations of "forward and backward" as it would for a time parameter.

If $w_{i,j}$ denotes the limiting probability of being in state i, j, reversibility implies that

$$b_{i+1,j}\, w_{i+1,j} = a_{ij}\, w_{ij}$$

$$d_{i,j+1}\, w_{i,j+1} = c_{ij}\, w_{ij}$$

$$g_{i+1,j+1}\, w_{i+1,j+1} = e_{ij}\, w_{ij}$$

$$h_{i+1,j-1}\, w_{i+1,j-1} = f_{ij}\, w_{ij} \tag{3}$$

Hence for a fixed value of j

$$w_{i+1,j} = \frac{a_{ij}}{b_{i+1,j}}\, w_{ij}$$

$$= \frac{a_{ij}}{b_{i+1,j}} \frac{a_{i-1,j}}{b_{ij}} \cdots \frac{a_{0j}}{b_{1,j}}\, w_{0j}$$

Varying j leads to

$$w_{ij} = \frac{a_{i-1,j}}{b_{ij}} \cdots \frac{a_{0j}}{b_{1,j}} \frac{c_{0j-1}}{d_{0j}} \cdots \frac{c_{00}}{d_{01}}\, w_{00} \tag{4}$$

CONDITIONS FOR POLYNOMIAL FACTORS

In the univariate case, necessary and sufficient conditions for polynomial factors (given that the process is reversible) were that (1) polynomials belong to $L^2(R,w)$; that is, $\Sigma\, w_i\, i^n < \infty$, and (2) Ai^n is a polynomial of degree n in i for each n.

When we try to rewrite the second condition for bivariate distributions, we run into problems because no natural ordering for terms $i^n j^m$ is known. In the univariate case, term i^n is of greater order than i^{n-1} but is $i^n j^m$ of greater order than $i^{n-1} j^{m+1}$? Terms could be ordered according to the sum of their powers, but this soon becomes unwieldy. So whereas for the univariate case we were able to deduce the form of transition probabilities, this is no longer easy. Instead we assume that transition probabilities are linear because this led to suitable univariate models, that is

$$a_{ij} = a_0 + a_1 i + b_2 j$$

$$b_{ij} = b_0 + b_1 i + b_2 j$$

etc. As the probability of going "backward" from $i = 0$ (or $j = 0$) to $i = -1$ (or $j = -1$) must be zero, $b_0 = 0 = b_2$ and $b_1 > 0$.

Taking account of all these "boundary" conditions leads to the following equations

$$a_{ij} = a_0 + a_1 i + a_2 i \qquad a_0 > 0$$

$$b_{ij} = b_1 i \qquad b_1 > 0$$

$$c_{ij} = c_0 + c_1 i + c_2 j \qquad c_0 > 0$$

$$d_{ij} = d_2 j \qquad d_2 > 0$$

$$e_{ij} = e_0 + e_1 i + e_2 j \qquad e_0 > 0$$

$$f_{ij} = f_2 j \qquad f_2 > 0$$

$$h_{ij} = h_1 i \qquad h_1 > 0$$

$$g_{ij} = 0$$

From reversibility, (Eq. 3)

$$e_{ij} = \frac{g_{i+1,j+1}\ w_{i+1,j+1}}{w_{ij}} = 0$$

The probability of going from (i,j) to $(i + 1, j + 1)$ must be the same passing either by $(i + 1, j)$ or by $(i, j + 1)$.

For the first pathway,

$$w_{i+1,j} = \frac{[c_0 + c_1 (i + 1) + c_2 j]}{d_2 (j + 1)} \cdot \frac{(a_0 + a_1 i + a_2 j)}{b_1 (i + 1)} w_{ij}$$

whereas for the second

$$w_{i,j+1} = \frac{[a_0 + a_1 i + a_2 (j + 1)]}{b_1 (i + 1)} \cdot \frac{(c_0 + c_1 i + c_2 j)}{d_2 (j + 1)} w_{ij}$$

Because this is true identically for all i,j

$$a_0 = (a_2/c_1)/c_0$$

$$a_1 = (a_2/c_1) \cdot c_1 = c_1$$

$$a_2 = (a_2/c_1)/c_2 \qquad \therefore c_1 = c_2 \tag{5}$$

A similar argument applied to transition from (i,j) to $(i + 1, j - 1)$ either going diagonally or via $(i, j - 1)$, leads to the condition

$$h_1 = \frac{f_2\, b_1\, c_1}{a_1\, d_2} \tag{6}$$

Consequently the system of transition probabilities reduces to

$$a_{ij} = a_0 + ai + aj \qquad a_0 > 0$$

$$b_{ij} = bi \qquad b > 0$$

$$c_{ij} = \frac{c\, a_0}{a} + ci + cj \qquad c_0 = \frac{c\, a_0}{a} > 0$$

$$d_{ij} = dj \qquad d > 0$$

$$f_{ij} = fj \qquad f > 0$$

$$h_{ij} = \frac{c\,f\,b\,i}{a\,d}$$

Because $c_0 = (c\,a_0/a) > 0$, a and c are of the same sign. So three cases to be considered, a and c both positive, both zero, or both negative.

In the univariate case, an isofactorial model with a quadratic expression for transition probabilities also was established. Probably a similar bivariate model also exists. However, as equations resulting from reversibility conditions are complicated, this possibility has not been pursued yet.

We now consider three cases (a and c both positive, both zero and both negative) in detail and show that, as could be guessed by analogy with the univariate case, these have the multivariate negative binomial (or negative multinomial), Poisson, and multinomial distributions as their limiting distributions. So we see that the two variables i and j both have either an infinite number of states or a finite number. The mixed case (i infinite and j finite) does not seem to arise.

BIVARIATE DISTRIBUTIONS

$a > 0$, $c > 0$ Negative Multinomial Distribution

In this case

$$a_{ij} = a_0 + ai + aj \text{ etc.} \qquad a_0 > 0,\ a > 0$$

From (4), the limiting distribution (for j fixed) is

$$\begin{aligned} W_{nj} &= \left(\frac{a_0 + ai}{b}\right)\left(\frac{a_0 + aj + a}{2b}\right) \cdots \left(\frac{a_0 + aj + (n-1)a}{n\,b}\right) w_{0j} \\ &= \left(\frac{a}{b}\right)^n \left(\frac{a_0}{a} + j\right)\left(\frac{a_0}{a} + j + 1\right) \cdots \left[\frac{a_0}{a} + j + (n-1)\right] \frac{w_{0j}}{n} \\ &= \left(\frac{a}{b}\right)^n \frac{\Gamma(\nu + n)}{\Gamma(\nu)\,n!}\, w_{0j} \end{aligned}$$

where $\nu = [(a_0/a) + j]$.

Applying the same reasoning to the second index gives

$$w_{nm} = \left(\frac{a}{a}\right)^n \left(\frac{c}{d}\right)^m \left[1 - \frac{a}{b} - \frac{c}{d}\right]^\gamma \frac{\Gamma(\gamma + n + m)}{\Gamma(\gamma)\,n!\,m!} \tag{7}$$

where $\gamma = a_0/a = c_0/c$.

This is a negative multinomial distribution.

$a = 0 = c$ Poisson Distribution

The limiting distribution (for j fixed) is

$$w_{nj} = \left(\frac{a_0}{b}\right)^n \frac{w_{0j}}{n!}$$

and (varying j)

$$w_{nm} = \left(\frac{a_0}{b}\right)^n \left(\frac{c_0}{d}\right)^m \frac{w_{00}}{n!\, m!}$$

Because

$$\sum_n \sum_m w_{nm}\, w_{00} = 1 = e - (a_0/b) - (c_0/d)$$

Hence

$$w_{nm} = \left(\frac{a_0}{b}\right)^n \frac{e^{-a_0/b}}{n!} \left(\frac{c_0}{d}\right)^m \frac{e^{-c_0/d}}{m!} \tag{8}$$

This bivariate distribution is of no further interest because it can be split into two independent Poisson distributions which can be treated separately.

$a < 0, c < 0$ Multinomial Distribution

Because all transition probabilities must be positive, only a finite number of states ($i = 0, 1, \ldots, N$) exist. Transition probabilities therefore can be rewritten as

$$a_{ij} = a(N - i - j)$$

where $a_{ij} = 0$ when $i + j = N$. The range of possible states (i, j) is illustrated (Figure 1).

The limiting distribution (for fixed j) is

$$w_{nj} = \left(\frac{a}{b}\right)^n \frac{N - j - n + 1) \cdots (N - j)}{n\,!}$$

$$w_{0j} \qquad n \leq N - j \qquad \text{and}$$

$$w_{nm} = \left(\frac{a}{b}\right)^n \left(\frac{c}{d}\right)^m \frac{N(N-1) \cdots (N - m - n + 1)}{n!\, m!} w_{00}$$

$$= \left(\frac{a}{b}\right)^n \left(\frac{c}{d}\right)^m \left(1 - \frac{a}{b} - \frac{c}{d}\right)^{N-m-n} \frac{N!}{n!m!(N - m - n)!} \tag{9}$$

This is the multinomial distribution.

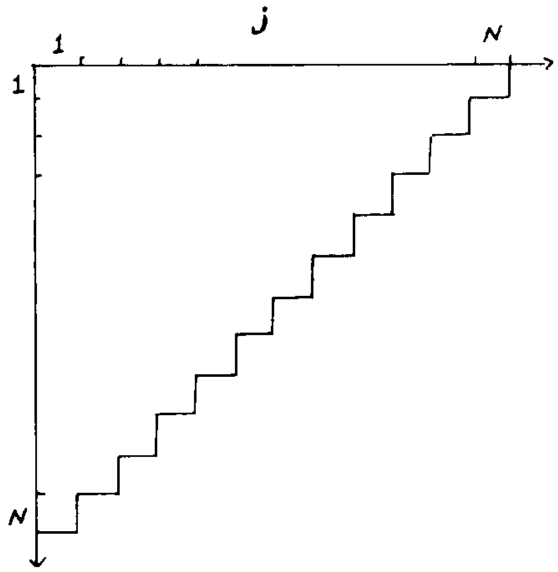

Fig. 1. Range of possible states (i, j) for the process with $a < 0$, $c < 0$.

POINT-POINT DISTRIBUTION

In previous work on isofactorial models, the objective was to find an isofactorial representation for the bivariate, point-point (or block-block), distribution for the case where point grades had a specified univariate distribution. In the case being considered here, the "marginal" distribution of point data is already a bivariate distribution and so the corresponding point-point (or block-block) distribution is quadrivariate. Clearly, this is going to lead to mathematical and notational problems.

In the next section, we see how the stochastic process associated with the negative multinomial distribution can be developed and factorized.

NEGATIVE MULTINOMIAL PROCESS

Using the same notation as before

$$a_{ij} = a[(a_0/a) + i + j]$$

$$= a(\gamma + i + j)$$

$$b_{ij} = b_i$$

$$c_{ij} = c(\gamma + i + j)$$

$$d_{ij} = d_j$$

We now use the same line of reasoning employing generating functions as was developed for the univariate case (Matheron, 1975b). In many cases, the generating functions involve a variable t. Because stochastic processes originally developed from time series, the variable will be called "time," even though our assumption of reversibility would exclude many time-dependent processes.

The probability generating function for this distribution is

$$\sum_n \sum_m w_{nm} s_1^n s_2^m = \sum_n \sum_m \left(\frac{a}{b}\right)^n \left(\frac{c}{d}\right)^m \left(1 - \frac{a}{b} - \frac{c}{d}\right)^\gamma \frac{\Gamma(\gamma + n + m)}{n!\mathrm{m}!\Gamma(\gamma)} s_1^n s_2^m$$
$$= \left(1 - \frac{a}{b} - \frac{c}{d}\right)^\gamma \left[1 - \frac{as_1}{b} - \frac{cs_2}{d}\right]^{-\gamma} \tag{10}$$

Let $G_{ij}(s_1, s_2, t)$ be the conditional probability generating function; that is

$$G_{ij}(s_1, s_2, t) = \sum_k \sum_l W_{ij:kl}(t) s_1^k s_2^l$$

where $w_{ij:kl}(t)$ denotes the stationary probability of going from state (i, j) to (k, l) in time t. If A denotes the infinitesimal generator, then

$$\frac{\partial G_{ij}}{\partial t}(s_1, s_2, t) = AG_{ij} \tag{11}$$

We now suppose that $b_{ij}(s_1, s_2, t)$ is of the form

$$G_{ij}(s_1, s_2, t) = \{V_t(s_1, s_2)\}^i \{V_t(s_1, s_2)\}^j \{H_t(s_1, s_2)\}^\gamma \tag{12}$$

A simple and intuitive way of seeing what this equation represents is to consider the process as a bivariate birth and death process concerning two types of bacteria (Type I and II). At the outset, Type I has i and Type II has j individuals. Descendants of any one of the i Type I bacteria after a time t can be described (probablistically) by the generating function $U_t(s_1, s_2)$. This is raised to the power i to give the progeny of all i Type I bacteria after time t; similarly for $[V_t(s_1, s_2)]^j$. The other term $[H_t(s_1, s_2)]^\gamma$ accounts for the progeny of bacteria which were generated spontaneously in that time. For more details on generating functions applied to birth and death processes, see Feller (1968, Chap. XI).

Return to Eq. 11 which can be rewritten as

$$\frac{1}{bG_{ij}} \frac{\partial G_{ij}}{\partial t} = \frac{1}{G_{ij}} AG_{ij}$$

Substituting (12) for G_{ij} gives

$$\frac{\gamma}{H}\frac{\partial H}{\partial t} + \frac{i}{V}\frac{\partial U}{\partial t} + \frac{j}{V}\frac{\partial V}{\partial t} = (\gamma + i + j)\,\{a(U - 1) + c(V - 1)\}$$
$$+ bi[(1/U) - 1] + dj[(1/V) - 1] \tag{13}$$

At time $t = 0$,

$$H_0(s_1, s_2) = 1$$
$$U_0(s_1, s_2) = s_1$$
$$V_0(s_1, s_2) = s_2$$

From this, equating terms, we obtain

$$\frac{1}{H}\frac{\partial H}{\partial t} = a(U - 1) + c(V - 1) \tag{14}$$

$$\frac{1}{U}\frac{\partial U}{\partial t} = a(U - 1) + c(V - 1) + b[(1/U) - 1] \tag{15}$$

$$\frac{1}{V}\frac{\partial V}{\partial t} = a(U - 1) + c(V - 1) + d[(1/V) - 1] \tag{16}$$

The solution to the first of these equations is

$$H_t(s_1, s_2) = \exp\left\{- \int_0^t a(U_\tau - 1) + c(V_\tau - 1)\, d\tau\right\}$$

Putting $F_t(s_1, s_2) = [H_t(s_1, s_2)]^{-1}$ and substracting (14) from (15) gives

$$\frac{1}{U}\frac{\partial U}{\partial t} + \frac{1}{F}\frac{\partial F}{\partial t} = b\left(\frac{1}{U} - 1\right)$$

Because $U_0(s_1, s_2)\, F_0(s_1, s_2) = s_1$, the general solution to this equation is

$$U_t(s_1, s_2)\, F_t(s_1, s_2) = s_1 e^{-bt} + b \int_0^t e^{-d(\tau - t)}\, F_\tau(s_1, s_2)\, d\tau$$

Similarly

$$V_t(s_1, s_2)\, F_t(s_1, s_2) = s_2 e^{-dt} + d \int_0^t e^{-d(\tau - t)}\, F_\tau(s_1, s_2)\, d\tau$$

Taking Laplace transforms of these two equations gives

$$U\tilde{F}(\nu) = \frac{s_1}{b + \nu} + \frac{b}{b + \nu}\tilde{F}(\nu) \tag{17}$$

$$V\tilde{F}(\nu) = \frac{s_2}{d + \nu} + \frac{d}{d + \nu}\tilde{F}(\nu) \tag{18}$$

where $\widetilde{UF}(\nu)$ is the Laplace transform of $U_t F_t$.

Moreover from (14)

$$-(\partial F/\partial t) - [a(U - 1) + c(V - 1)]F \tag{19}$$

On taking Laplace transforms and substituting results given in (17) and (18) into (19), we obtain

$$1 - \nu\tilde{F}(\nu) = \frac{as_1}{b + \nu} + \frac{cs_2}{d + \nu} - \nu\left[\frac{a}{b + \nu} + \frac{c}{d + \nu}\right]\tilde{F}(\nu)$$

Hence,

$$\tilde{F}(\nu) = \frac{(b + \nu)(d + \nu) - as_1(d + \nu) - cs_2(b + \nu)}{\nu Q(\nu)} \quad \text{where}$$

$$\begin{aligned} Q(\nu) &= \nu^2 + \nu(b + d - a - c) + bd - ad - cb \\ &= \nu^2 + \nu[(b - a) + (d - c)] - (b - a)(d - c) - ac \end{aligned}$$

We already have seen that for the bivariate distribution to be ergodic, b must be greater than a and d greater than c. Consequently, this quadratic equation $Q(\nu) = 0$ has two real positive roots C_0 and C_1. After some simple but tedious algebra

$$\tilde{F}(\nu) = \frac{1 - (as_1/b) - (cs_2/d)}{1 - (a/b) - (c/d)} + \frac{B_0(s_1, s_2)}{\nu + C_0} + \frac{B_1(s_1, s_2)}{\nu + C_1} \quad \text{where}$$

$$B_0(s_1, s_2) = \frac{(b - C_0)(d - C_0)}{C_0(C_1 - C_0)}\left[1 - \frac{as_1}{b - C_0} - \frac{cs_2}{d - C_0}\right]$$

$$B_1(s_1, s_2) = \frac{(b - C_1)(d - C_1)}{C_1(C_0 - C_1)}\left[1 - \frac{as_1}{b - C_1} - \frac{cs_2}{d - C_1}\right]$$

On inverting the Laplace transform,

$$F_t(s_1, s_2) = \frac{1 - (a/b)s_1 - (c/d)s_2}{1 - (a/b) - (c/d)} + B_0(s_1, s_2)\, e^{-C_0 t} + B_1(s_1, s_2)\, e^{-C_1 t} \tag{20}$$

Expressions for $U\tilde{F}(\nu)$ and $V\tilde{F}(\nu)$ can be found by substitution into (17) and (18).

$$\begin{aligned} \widetilde{UF}(\nu) &= \frac{s_1 + b\tilde{F}(\nu)}{b + \nu} \\ &= \frac{1 - (a/b)s_1 - (c/d)s_2}{1 - (a/b) - (c/d)} + \frac{bB_0}{(b - C_0)(\nu + C_0)} + \frac{bB_1}{(b - C_1)(\nu + C_1)} \end{aligned}$$

Hence,

$$U_tF_t = \frac{1 - (a/b)s_1 - (c/d)s_2}{1 - (a/b) - (c/d)} + \frac{bB_0 e^{-C_0 t}}{c - C_0} + \frac{bB_1 e^{-C_1 t}}{c - C_1} \tag{21}$$

Similarly for $V_t F_t$,

$$V_t F_t = \frac{1 - (a/b)s_1 - (c/d)s_2}{1 - (a/b) - (c/d)} + \frac{dB_0 e^{-C_0 t}}{d - C_0} + \frac{dB_1 e^{-C_1 t}}{d - C_1} \tag{22}$$

Having now obtained expressions for functions $U_t(s_1, s_2)$, $V_t(s_1, s_2)$, and $H_t(s_1, s_2)$, which together make up $G_{ij}(s_1, s_2, t)$, we proceed to find an expression for the quadrivariate equivalent of the point-point distribution. The generating function $G(s_1, s_2, \tau_1, \tau_2)$ of this process can be deduced from the generating function for the bivariate negative binomial process

$$\left(1 - \frac{a}{b} - \frac{c}{d}\right)^{\gamma} \left(1 - \frac{a\tau_1}{b} - \frac{c\tau_2}{d}\right)^{-\gamma}$$

by substituting generating functions U_t, V_t, and H_t in appropriate places. This gives

$$\left\{\left(1 - \frac{a}{b} - \frac{c}{d}\right) H\right\}^{\gamma} \left\{1 - \frac{a\tau_1 U}{b} - \frac{c\tau_2 V}{d}\right\}^{-\gamma}$$

Replacing H_t by F_t^{-1} gives

$$\left(1 - \frac{a}{b} - \frac{c}{d}\right)^{\gamma} \left[F - \frac{a\tau_1}{b} UF - \frac{c\tau_2}{d} VF\right]^{-\gamma}$$

On substituting formulas for F, UF, and VF we obtain an explicit expression for $G(s_1, s_2, \tau_1, \tau_2)$

$$[K(1, 1)]^{2\gamma}[K(s_1, s_2)\, K(\tau_1, \tau_2) - (1 - K(1, 1))\, r_0 L_0(s_1, s_2)\, L_0(\tau_1, \tau_2)\, e^{-C_0 t}$$

$$- (1 - K(1, 1))\, r_1 L_1(s_1, s_2)\, L_1(\tau_1, \tau_2)\, e^{-C_1 t}]^{-\gamma}$$ where

$$K(s_1, s_2) = 1 - \frac{as_1}{b} - \frac{cs_2}{d}$$

$$L_0(s_1, s_2) = 1 - \frac{as_1}{b - C_0} - \frac{cs_2}{d - C_0}$$

$$L_1(s_1, s_2) = 1 - \frac{as_1}{b - C_1} - \frac{cs_2}{d - C_1}$$

$$r_0 = \frac{K(1, 1)\,(b - C_0)\,(d - C_0)}{K(s_1, s_2)\, C_0(C_1 - C_0)} \geq 0$$

$$r_1 = \frac{K(1, 1)\,(b - C_1)\,(d - C_1)}{K(s_1, s_2)\, C_1(C_0 - C_1)} \geq 0$$ and

$$r_0 + r_1 = 1$$

Having obtained the expression for $G(s_1, s_2, \tau_1, \tau_2)$, we proceed to see whether this can be expanded into an isofactorial form. This is done in two steps. We rewrite the expression for $G(s_1, s_2, \tau_1, \tau_2)$ in a form suitable for a negative binomial expansion

$$G(s_1, s_2, \tau_1, \tau_2) = \{1 - p(x_0 + x_1)\}^{-\gamma}$$

where $p = 1 - K(1, 1)$

$$x_0 = \frac{r_0 L_0(s_1, s_2)\, L_0(\tau_1, \tau_2)\, e^{-C_0 t}}{K(s_1, s_2)\, K(\tau_1, \tau_2)}$$

$$x_1 = \frac{r_1 L_1(s_1, s_2)\, L_1(\tau_1, \tau_2)\, e^{-C_1 t}}{K(s_1, s_2)\, K(\tau_1, \tau_2)}$$

Expanding this gives

$$G(s_1, s_2, \tau_1, \tau_2) = \sum_0^\infty \frac{\Gamma(\gamma + n)}{\Gamma(\gamma) n!} p^n \sum_{k=0}^{n} C_k x_0^k x_1^{n-k}$$

The generating function is not our final objective. We want the quadrivariate distribution with probability $p_{ij:kl}$ in an isofactorial form. For discrete univariate marginal distributions $p(i)$, an isofactorial representation of the bivariate point–point distribution $p(i, k)$ is of the form

$$p(i, k) = \sum_n T_n \chi_n(i)\, \chi_n(k)\, p(i)\, p(k)$$

Similarly, when the point distribution is a bivariate $p(i, j)$ isofactorial representation for the quadrivariate distribution is

$$p(i, j\!:\!k, l) = \sum T_n \chi_n(i, j)\, \chi_n(k, l)\, p(i, j)\, p(k, l)$$

What we need are factors $\chi_n(i, j)$ and coefficients T_n. The generating function $G(s_1, s_2, \tau_1, \tau_2)$ gives the generating function for factors as well as eigenvalues of the infinitesimal generator A which allow us to find T_n. Eigenvalues appear in the exponential term

$$e^{-[kC_0 + (n-k)C_1]t}$$

So they are

$$[kC_0 + (n - k)C_1]$$

We obtain the generating function of factors by considering term $x_0^k x_1^{n-k}$. The generating function $G_{n,k}(s_1, s_2)$ of the factors comes from regrouping all terms in s_1 and s_2 which occurred in $x_0^k x_1^{n-k}$. This gives

$$G_{n,k}(s_1, s_2) = \frac{[K(1, 1)]^\alpha\, [L_0(s_1, s_2)]^k\, [L_1(s_1, s_2)]^{n-k}}{[K(s_1, s_2)]^{n+\alpha}}$$

This function has been normalized so that factor $\chi_{n,k}(0, 0)$ will be equal to 1. This normation will not be valid if $b = d$; the index r_0 is zero and $C_0 = C_1$.

This generating function easily is shown to satisfy equation

$$AG = -\lambda G$$

for $\lambda = kC_0 + (n - k)C_1$, and also that

$$AG = [a(s_1 - 1) + c(s_2 - 1)]\,[\alpha G + s_1(\partial G/\partial s_1) + s_2(\partial G/\partial s_2)]$$
$$+ b(1 - s_1)\,(\partial G/\partial s_1) + d(1 - s_2)\,(\partial G/\partial s_2)$$

CONCLUSION

Results presented in this paper show that bivariate isofactorial models can be developed. We hope that this will stimulate interest in problems of predicting recovery and recovered grades after the initial separation. This could be a considerable aid to mine planners.

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