PRECISION OF EXPLORING A STRATIFIED FORMATION BY BOREHOLES WITH RIGID SPACING—APPLICATION TO A BAUXITE DEPOSIT

G. MATHERON

SUMMARY

THIS ARTICLE consists of two parts, a recapitulation of the theoretical formula relative to boreholes with rigid spacing and an experimental verification in the case of a bauxite deposit.

1. *Theoretical Formulae*

The statistical population is considered as constituted by the assay values x of boreholes in infinite numbers which can theoretically be drilled into the deposit. No assumption is made regarding the law of statistical distribution of these assay values. The assay values x_1 and x_2 of two boreholes at a distance d from each other are not independent. The smaller the distance d is, the less will be, on an average, the difference between the assay values x_1 and x_2. This correlation expresses in a statistical sense the existence of a certain degree of continuity of the mineralization and may be represented by a variogram, i.e. the curve of $\sigma_d^2 = f(d)$ which gives the variance of the difference $(x_1-x_2)^2$ as a function of the distance d.

The variogram can often be represented by a logarithmic formula $f(d) = A+4\alpha \log d$, where α is the absolute scatter. This expression cannot be used any more when the values for the distance d are very small. If the spacing of the boreholes is, however, great compared with the thickness of the bed, the behaviour of $f(d)$ around $d = 0$ is of no practical importance.

The variance of sampling is the variance which defines the precision with which the assay value of a deposit, explored by n boreholes with a spacing d, is known and is put in the form

$$\sigma^2 = \frac{1}{2n} f\left(\frac{d}{2.91}\right).$$

2. *Experimental Verification*

The African bauxite deposit of Mehengui was explored by 300 boreholes with a 50 m spacing. Moreover, two squares of 50×50 were explored for every 121 boreholes with a 5 m spacing.

The variogram was constructed on the basis of 29 experimental points, included between $d = 5$ and $d = 500$. The distinction between the pairs in the direction NS and EW and NE–SW and NW–SE does not reveal any anisotropy. The values relative to each of the two 50×50 squares are in remarkable parallelism which indicates that the small irregularities of the curve correspond to a real phenomenon. All the experimental points are approximately in alignment in a logarithmic diagram. The deviations are not caused by simple fluctuations but their amplitude is fairly small, so that the theoretical formulae remain usable.

Among the 300 boreholes with 50 m spacing it is possible to extract an exploration scheme of spacing 200 in 16 different ways. We have studied the distributions of the mean and of the variance of these 16 schemes. The scatter of the mean values, in particular, enables a good estimate to be made of the variance of sampling of spacing 200. The experimental value obtained is in excellent agreement with the value deduced from the theoretical formulae:

	Theoretical prediction	Experimental value
Variance of sampling σ^2	0.62	0.60
Possible error $\pm 2\sigma$	$\pm$ 1.57	$\pm$ 1.55

I. GENERAL REMARKS ON THE PRECISION OF A SPACING OF EXPLORATION

(a) *Conception of Variance—Variance of Sampling*

It is theoretically possible to sink into a deposit an infinite number of boreholes or shafts. The whole of the assay values x_i, which all these boreholes or shafts would produce, constitutes a *statistical population*. It is convenient to characterize such a population by two parameters:

—an arithmetical *mean*

$$m = \frac{1}{n}\sum x_i \text{ (where } n = \text{number of boreholes)}$$

which is no other than the mean assay value of the deposit.

—a *variance* or mean quadratic deviation

$$\sigma^2 = \frac{1}{n}\sum (x_i - m)^2$$

which measures the degree of scatter of the individual values around their mean value.

The square root of the variance is the *standard deviation* σ. If the individual values are distributed according to the log-normal law (bell-shaped curve), 95 per cent of them are included in the interval $m \pm 2\sigma$. Any value, chosen at random, has 95 chances out of 100 to fall within this interval. The interval $\pm 2\sigma$ gives therefore a practical measure for the scope of possible variation.

When a deposit has been explored by a limited number n of boreholes the mean m' of the assay values of these boreholes is generally not exactly equal to the real mean assay value m of the deposit. We can imagine that it would be possible to drill in an infinite number of different ways the same number n of boreholes with the same regular spacing, with the possible different schemes deduced from each other by transfer. We can also imagine that each of these schemes would give a different arithmetical mean m'. On these grounds m' appears as a random variable and the whole of the mean values m' corresponding to all the possible holes drilled constitutes a statistical population. This population can be characterized in turn by two parameters:

(1) a mean value which is no other than the true assay value of the deposit,

(2) a variance which we shall call *the variance of sampling* σ^2_E.

This variance of sampling characterizes the scatter of the possible estimates of the mean assay value for a given number of boreholes and a given spacing, in other words, *it measures the precision of a scheme of exploration.*

It is interesting to consider the standard deviation of sampling σ_E. In fact, even if the assay values of the boreholes are not rigorously distributed according to a normal law, the arithmetical means m' have a distribution which converge towards the normal type when the number n increases. In the case of bauxite deposits, we are practically assured of normality. The result of this is that the mean assay value furnished by any scheme of holes drilled with a total number of n boreholes at a fixed spacing has 95 chances out of 100 to fall in the interval $m \pm 2\sigma_E$.

In reality, m is not known. We can understand, however, that there are also nearly 95 chances out of 100 that the real mean assay value m differs from the value m' furnished by the boreholes only by a quantity of less than $2\sigma_E$ in absolute value.

Practically, this means that we can take as limit of the error in mean assay value twice the value, i.e. $+2\sigma_E$ *of the standard deviation of sampling.*

(*b*) *Theoretical Expression of the Variance of Sampling—The Variogram*

Mathematically speaking, the assay values x_1 and x_2 which would result from two boreholes drilled at a distance d from each other cannot be considered as independent data. It is understandable that if the distance d is small and if the borehole *l* as produced a high assay value x_1, the borehole 2 drilled at the distance d has more chances of producing a high assay value than a borehole drilled at random. Two adjoining boreholes will produce, on an average, assay values which are less different from each other than any two other boreholes. This correlation, in the statistical sense, between the assay values of adjacent boreholes signifies nothing other than the existence of a certain *degree of continuity* of the mineralization. It is interesting to try to express this degree of continuity in figures which must obviously differ depending on the types of deposit. Let us simply indicate here that it is convenient to represent this continuity relationship by a curve which we shall call a *variogram.*

If we consider all the pairs x_1 and x_2 of the assay values of possible boreholes drilled in the deposit at the distance d from each other, the variable x_1-x_2 is characterized by a mean which is zero and a variance of:

$$\sigma_d^2 = \frac{n}{1} \sum (x_1-x_2)^2$$

which gives a measure of the mutual dependence of these two assay values. The variogram is the curve which represents this variance as a function of the distance d, i.e.

$$\sigma_d^2 = f(d). \tag{1}$$

We expect that $f(d)$ is a function which rises with the distance d since the nearer the boreholes are to each other, the less their assay values should differ, on an average. At the limit, i.e., for $d = o$ $f(d)$ must become zero since the two boreholes coincide. The behaviour of the function near the origin has, however, no great importance from the practical point of view. In the range of customary usage, the variogram can often be represented in a very acceptable fashion by an equation of the form

$$f(d) = A + 4\alpha \log d. \tag{2}$$

The parameter α or absolute scatter, characterises the degree of discontinuity of the mineralization. It is an intrinsic parameter, the theoretical significance of which is closely connected with the principle of similarity and is of great significance for the metallogenesis.

Without insisting here on these theoretical implications, we regard this relationship as representing a good interpolation of the experimental data.

We can demonstrate that, if that relationship is verified, the variance of sampling which fixes the precision with which the assay value of a deposit explored by n boreholes drilled in a square spacing of dimension d, is given by the formula:

$$\sigma^2 = \frac{\sigma 2d}{2n} - \frac{2.14\alpha}{n} = \frac{1}{2n} f\left(\frac{d}{2.91}\right). \tag{3}$$

It is not surprising that the variance of sampling takes a form of $1/n$ and is closely connected with the variogram. In fact, the possible error comes from the fact that the assay value x_i of each borehole differs from the real assay value m_i of its "polygon of influence". The total error is of the form

$$\frac{1}{n} \sum (x_i - m_i).$$

If the differences $(x_i - m_i)$ can on an average be considered as independent of each other, and if h^2 is the variance of $(x_i - m_i)$ the variance of sampling must be equal to h^2/n. Therefore:

The variance is indeed a function of $1/n$ which means that *at the same spacing* a four times larger deposit (explored by four times more boreholes) is known with twice the precision.

It is proportional to h^2 which represents a sort of semi-mean of the values of the variogram for the values of distance of less than d. It is, therefore, not surprising to find a formula of the type (3). In fact the numerical value of 2.91 for the factor of d is the only one to depend on the logarithmic nature of the variogram.

The interest of the formula (3) is that the variogram $f(d)$ is directly accessible to experiment.

2. APPLICATION TO THE BAUXITE DEPOSIT OF MEHENGUI

The Data

The stratified bauxite deposit of Mehengui forms a continuous, not very thick bed. It has been explored by 300 effective boreholes, drilled at a square spacing of 50 m. Moreover, two 50×50 squares, designated in the following by the names Zone I and Zone II, were each explored by 121 boreholes at 5 m spacing in an aim to test the homogeneity of the assay values. Each borehole is characterized by its Al_2O_3 content taken between hanging wall and foot wall.

Aim of this Study

The object of this study is principally to verify from the experimental data the validity of the law $\sigma_{xy} = A - 2\alpha \log d$ which gives the covariance xy of the assay values of boreholes a distance of d apart. It is possible for the Mehengui deposit to calculate numerically such covariances for values of the distance d lying between 5 and 500 m, i.e. an interval of variation of 1 to 100.

To eliminate the boundary effect, we shall calculate numerically, not the covariances themselves, but the variances σ_d^2 of the differences $(x-y)$ of the assay values x and y of boreholes a distance of d apart. We have:

$$\sigma_d^2 = 2(\sigma^2 - \sigma_{xy}) = B + 4\alpha \log d$$

where σ^2 is the variance of the assay values of the boreholes. The expression σ_d^2 presents also the advantage that it is independent of the dimension of the deposit or of the portion of the deposit from which the data used for the calculation are derived. It is an intrinsic function of the distance d. It will consequently be possible to plot on the same graph the data derived from the 50 m spacing and from the two squares explored by 5 m spacing.

Study of the 50 m spacing. The distribution chart of the Al_2O_3 contents is reproduced as follows:

Values	No. of Boreholes	Values	No. of Boreholes
27	1	43	37
32	1	44	26
33	3	45	36
34	3	46	24
35	6	47	15
36	7	48	13
37	11	49	6
38	14	50	5
39	12	51	1
40	22	52	1
41	19	53	1
42	36		

We deduce therefrom:

Number of boreholes $N = 300$

Mean of assay values $m = 42.63$

Variance of assay values $= 15.30$.

The variances of the differences $(x-y)$ of boreholes at a distance d apart were calculated for 10 values of d from 50 to 500 m and separately for the EW and NS lines, see following table:

d	Number of data				Variance $\sigma\ d^2$
(unit 50 m)	EW	NS	EW	NS	EW + NS
1	267	266	23.7	26.0	25.1
2	245	243	26.4	24.6	25.5
3	223	233	26.6	28.4	27.6
4	199	222	30.2	26.8	28.5
5	180	204	28.8	31.4	30.1
6	159	185	31.0	31.0	31.0
7	138	173	35.3	29.1	31.9
8	118	161	30.5	25.7	27.8
9	95	147	30.5	30.5	29.5
10	76	120	31.6	27.8	29.2

The EW and NS variances are compatible with each other, bearing in mind the number of available data. A test χ^2 regarding the homogeneity of the EW and NS distribution of the differences of the order 1 gives $\chi^2 = 11.09$ for 10 degrees of liberty, i.e. $P = 0.35$ which is entirely admissible.

There is no anisotropy.

The same calculation was carried out for the differences taken separately regarding the NE–SW and NW–SE diagonals, as follows:

d (units 50 m)	Number n of data			Variance $\sigma^2 a$		
	NE–SW	NW–SE	Total	NE–SW	NW–SE	Total
1.41	249	248	497	27.10	21.75	24.45
2.83	224	216	440	33.3	29.8	31.5
4.24	204	189	393	29.0	30.3	29.7
5.65	182	160	342	31.8	28.6	30.3
7.06	161	129	290	31.4	33.6	32.4
8.49	137	100	237	34.9	31.8	33.7
9.90	119	73	192	38.2	33.8	37.0
11.30	103	52	155	28.3	27.6	28.3

Study of the 5 m spacing (Zones I and II)

	Zone I	Zone II
Number of Data	121	121
Mean	44.6	43.3
Variance	9.30	7.45

The two means are significantly different. The two variables are probably not like that but it is difficult to make an accurate test since the 121 data of each square are not independent.

The study of the σ_d^2 does not reveal any significant differences between the NS and the EW data or between those of NE–SW and NW–SE in Zone I or II. On the other hand, the global values are significantly different between Zone I and Zone II. The curve I shows the remarkable parallelism of the curves $\sigma_d^2 = f(d)$ for each of the two zones. It is impressive to find that every little irregularity of a curve which one would be tempted to attribute to a non-significant fluctuation has its exact counterpart on the other curve. Very characteristic is the variance of the slope (second derivative):

	1.41	2	2.82	3	4	4.25	5	5.65	6	7	8	8.5	9
Zone I	+	−	+	−	−	+	−	+	−	+	+	+	−
Zone II	+	−	+	−	−	+	−	+	−	+	−	+	−

The sign of these variations is the same in 12 out of 13 points. This means that these small irregularities of fortuitous appearance correspond to a real phenomenon. This is admittedly small enough in amplitude not to stand in the way of an interpolation by a continuous curve.

Zone I

d (unit 5 m)	Number of data			Variance σd^2		Total
	NS	EW	NS + EW	NS	EW	
1	110	110	220	12.65	11.6	12.12
2	99	99	198	14.7	11.55	14.62
3	88	88	176	17.8	14.3	16.05
4	77	77	154	20.0	15.6	17.80
5	66	66	132	22.5	18.4	20.45
6	55	55	110	21.2	22.9	22.05
7	44	44	88	24.0	17.4	20.70
8	33	33	66	24.3	18.9	21.60
9	22	22	44	33.8	25.1	29.45
10	11	11	22	32.1	27.8	29.95

It is hardly possible to infer a heterogeneity of the values relative to the EW and NS series. There is a horizontal isotropy in the Zone I for the 5 m spacing. It is the same in Zone II where we must not attach too much importance to the divergencies which become apparent starting from line 7. The latter remain admissible having regard to the small number of data.

Zone II

d (unit 5 m)	Number of data			Variances σd^2		Total
	NS	EW	NS + EW	NS	EW	
1	110	110	220	6.85	6.61	6.73
2	99	99	198	9.30	10.2	9.75
3	88	88	176	11.72	12.9	12.31
4	77	77	154	12.80	14.6	13.70
5	66	66	132	13.85	16.3	15.07
6	55	55	110	16.25	20.4	18.32
7	44	44	88	13.25	24.6	18.92
8	33	33	66	12.50	29.8	21.15
9	22	22	44	10.15	36.3	23.22
10	11	11	22	13.30	30.1	21.70

Finally, the values for σ_d^2 were calculated for each zone along the diagonals of 45°, without making any distinction between the NE–SW and NW–SE lines. (The distribution charts, established separately for each of the directions are practically identical, as was to be expected on account of the absence of any anisotropy.)

d (unit 5 m)	Number of data		Total	Variance σd^2		Total
	Zone I	Zone II		Zone I	Zone II	
1.41	200	200	400	11.91	7.28	9.59
2.83	162	162	324	15.60	10.20	12.90
4.24	128	128	256	17.60	11.21	14.40
5.65	98	98	196	20.90	13.34	17.12
7.06	72	72	144	20.00	16.38	18.19
8.49	50	50	100	24.01	21.62	22.83
9.90	32	32	64	25.85	24.6	25.22

Having acknowledged the absence of anisotropy we can plot on the same diagram the values for σ_d^2 from 5 to 5 m and from $5\sqrt{(2)}$ to $5\sqrt{(2)}$. The points 7.06 and 9.30 of this latter spacing will be identified with the points 7 and 10 of the former and the corresponding data, grouped together, are as follows:

d (unit 5 m)	Number of data			Variances σ'^2		Total
	Zone I	Zone II	Total	Zone I	Zone II	
1	220	220	440	12.12	6.73	9.42
1.41	200	200	400	11.91	7.28	9.60
2	198	198	396	14.62	9.65	12.18
2.83	162	162	324	15.60	10.20	12.90
3	176	176	352	16.05	12.31	14.18
4	154	154	308	17.80	13.70	15.75
4.24	128	128	256	17.60	11.21	14.40
5	132	132	264	20.45	15.07	17.76
5.65	98	98	196	20.90	13.34	17.12
6	110	110	220	22.05	18.32	20.18
7	160	160	320	20.35	17.65	18.95
8	66	66	132	21.60	21.15	21.37
8.49	50	50	100	24.04	21.62	22.83
9	44	44	88	29.45	23.22	26.33
10	54	54	108	27.70	23.40	25.55

These values are plotted in Fig. 1. The diagram shows clearly the close parallelism of the two curves relative to Zones I and II, a parallelism which goes a far as to reproduce in detail the fluctuations of the variation of slope. We concluded herefrom earlier that the small irregularities of the curve, in spite of their fortuitous appearance, should correspond to a real phenomenon, though of small amplitude. It appears also as if the two series of values are significantly different.

The Zone I in which the variance of 121 boreholes is 9.30 gives greater values than the Zone II where this variance is 7.45. The difference between

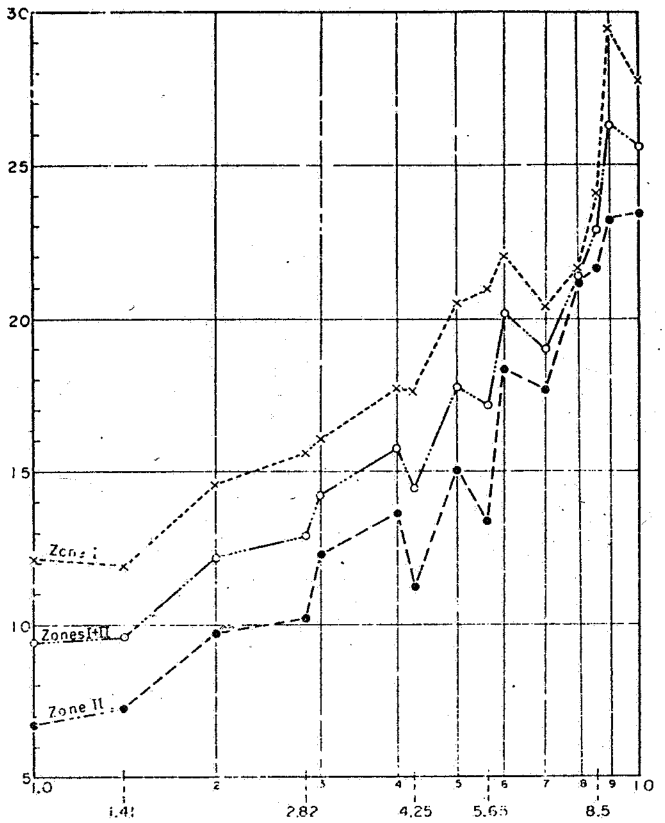

FIG. 1. Variogram $\sigma_d^2 = f(d)$ for the squares ZI and ZII.

the two series of values is nearly constant, and equal to twice (9.30–7.45) = 3.70 of the difference of the variances. We have also for each σ_d^2

$$\sigma_d^2 = 2(\sigma^2 - \sigma_{x,y})$$

where $\sigma_{x,y}$ is the covariance (*calculated in the square*, i.e. in the population of the 121 boreholes) of the boreholes at a distance d apart. It appears therefore that the curve giving this covariance as a function of d is nearly the same for each of the two squares. This is represented on Fig. 2.

Therefore, the differences of σ^2 between the two zones results solely from the difference of the variances σ^2 of the two populations. This difference, once it exists, has repercussions numerically over all the σ_d^2 values in spite of the identity of the covariances. The difference of the σ_d^2 is thus not more significant than that of the two variances (and we have seen that we were unable to affirm that it was so).

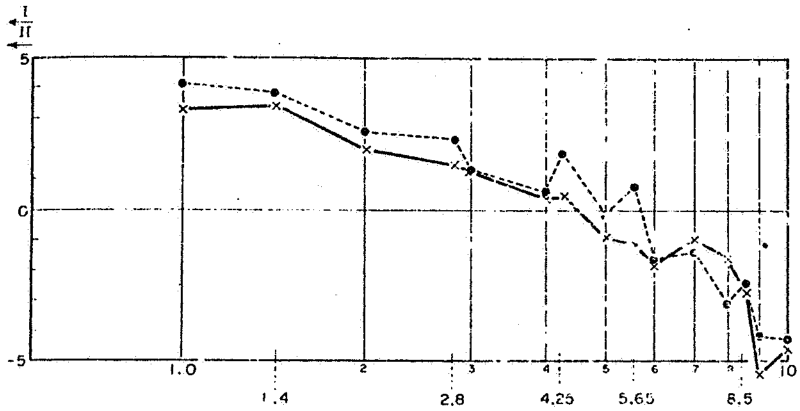

FIG. 2. Covariances σ_{xy} as function of the distance d between the boreholes (unit = 5m) for each of the squares ZI and ZII.

REPRESENTATION OF THE ENTIRE PHENOMENON

Figure 3 represents all the data regrouped in a single curve.

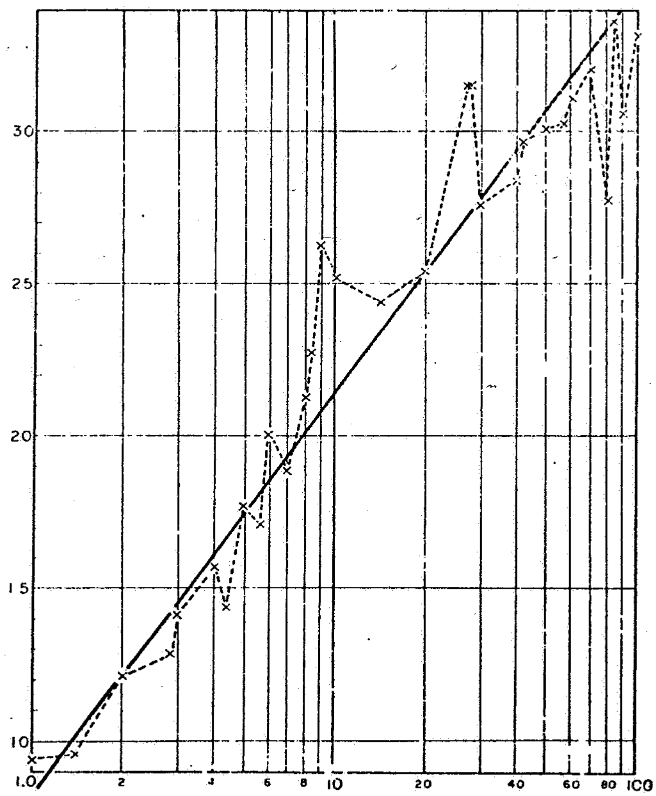

FIG. 3. Variogram $\sigma_d^2 = f(d)$.

DD

d (unit 5 m)	Number of data	Variance σd^2
1	440	9.42
1.41	400	9.60
2	306	12.18
2.83	324	12.90
3	352	14.18
4.24	308	15.75
4	256	14.4
5.65	264	17.76
5	196	17.12
6	220	20.18
7	320	18.95
8	132	21.37
8.49	100	22.83
9	88	26.33
10	641	25.20
14.1	497	24.45
20	488	25.50
28.2	440	31.50
30	456	27.60
40	421	28.50
42.5	393	29.70
50	384	30.1
56.5	342	30.3
60	344	31.0
70	601	32.1
80	279	27.8
84.9	107	33.7
90	242	30.5
100	398	33.2

The alignment in a logarithmic scale is on the whole satisfactory. The three anomalies for 9, 28, and 30 are almost significant.

Although we cannot affirm it, it is probable that they correspond to small real deviations. It remains nevertheless true that, on the whole, the phenomenon obeys a logarithmic law. The deviations which appear serious would almost disappear if we restored the arithmetic scale from 1 to 100. We believe that the logarithmic law

$$\sigma_d^2 = 7.9 + 5.76 \log d \text{ (Neperian logarithms, } d = 5\text{ m unit)}$$

can be used to make some predictions.

DISCUSSION OF ONE SPACING

Starting from the preceding logarithmic formula, we are now going to establish the law giving the precision at a fixed spacing. We shall then compare the precision really obtained at a spacing of 200 m with the theoretical precision which we have found.

1. *Theoretical Formula*

For a deposit explored by n boreholes with a spacing of a the variance of sampling is

$$\sigma_d^2 = \frac{\sigma u^2}{2n} - \frac{2.14a}{n} = \frac{1}{2n} f\left(\frac{a}{2.91}\right).$$

We have to divide the value of σ_E^2 for $a/2.91$ by $2n$. For example, for a spacing of 200 m (a = 40 in. 5 m units) we shall take a σ_d^2 corresponding to 40/2.91 = 13.75 with $\sigma_a^2 = 23.2$. Hence

$$\sigma_E^2 = \frac{23.2}{2n}$$

For the deposit of Mehengui which was explored by 300 boreholes with 50 m spacing, we should have with a 200 spacing

$$n = \frac{300}{16} = 18.7$$

and

$$\sigma_E^2 = \frac{23.2}{2\times 18.7} = 0.62$$

Hence $\begin{cases} \text{Standard deviation } \sigma_E = 0.785 \\ \text{Possible error (at 95\% level) } \pm 2\sigma_E = \pm 1.57. \end{cases}$

Such are the predictions of the theory which we are now going to compare with the experimental data.

2. *Experimental Verification*

Starting from the 300 boreholes drilled at a 50 m spacing, we can select 16 schemes of possible ways of drilling at a 200 spacing. For each of these schemes we have calculated the mean and the variance of the assay values. Since they are 16 in number, we can obtain a good approximation of the law of distribution of the estimate for the mean at a spacing of 200, therefore, a good estimate for σ_E^2.

The 16 values obtained for the mean are spaced out between 40.90 and 43.94, i.e. in an interval of 3 points. It is interesting to see that the number of useful boreholes varies between 15 and 22 which may give an idea of the precision with which the tonnage would be known on the basis of a 200 spacing. The variances are also similar and extremely scattered. The principal interest lies, however, in the distribution of the mean values.

No. of scheme	Number of data	Mean	Variance
1	19	43.58	13.05
2	22	43.54	10.60
3	19	41.79	16.05
4	19	43.00	7.90
5	22	42.73	13.80
6	22	42.86	16.40
7	20	40.90	9.24
8	19	42.73	13.10
9	19	43.94	17.60
10	20	43.15	9.84
11	18	42.11	15.75
12	19	41.79	12.90
13	15	42.93	30.60
14	16	42.75	23.40
15	15	41.60	33.60
16	16	42.31	11.40

Distribution of the mean values. This distribution is well characterized by the two customary parameters:

$$\text{mean } m = 42.59$$
$$\text{variance } \sigma_E^2 = 0.596.$$

The mean $m = 42.59$ is different from the mean of the 300 boreholes (42.63) because the 16 schemes include a variable number of boreholes. The variance σ_E^2 is remarkably close to the theoretical value, calculated earlier (0.62 against 0.596). This is not a fortuitous coincidence. If we had estimated this variance by the common process which consists of dividing the variance of the boreholes by $n = 18.7$, i.e. $15.30/18.7 = 0.82$, the value obtained would have been considerably higher. We have here, therefore, a good confirmation of the theories of the estimate based on the formula of the logarithmic covariance.

Distribution of variances. It is interesting to consider this distribution on a theoretical plane. The 16 variances form a population

$$\text{of mean } E(\sigma^2) = 15.95$$
$$\text{of variance } D^2(\sigma^2) = 54.5.$$

If the mean does not differ significantly from the general variance (15.30) of the 300 boreholes, the variance $D^2(\sigma^2)$ is, on the other hand, considerably higher than the variance $2/n(\sigma^4) = 32$ which we should have observed if our 16 sub-populations had been drawn at random in the deposit.

This rise is the logical counterpart of the fact that the variance $D^2(m)$ of the estimate of the mean is smaller than if the sub-populations were aleatory. The underlying reason is that the 18 elements of each population have a negative correlation between them. The mean values should be less scattered than those of the random sub-populations as one sub-population including an abnormally high value will have more chance of including an abnormally low value too. Inversely, a population short of high values will also have the tendency to be short of very low values. This explains the rise in scatter of the variances relating to random sub-populations.

If, therefore, the rigid spacing improves the estimate of the mean, it worsens the estimate of the variance. The corresponding theory is still to be put up. It is of great importance in the log-normal case for the Sichel valuator in which median and variance appear *together*. This valuator should be completely altered.

3. *Study of* 200 × 200 *squares*

Profiting from the grouping of the data effected for the experimental verifications of the preceding paragraph, we wanted to study also the distribution in the deposit of the assay values of 200 × 200 squares. We retained the 16 squares containing at least 13 boreholes, and studied the 16 corresponding sub-populations. The means and variances of these sub-populations are as follows:

Square	No. of boreholes	Mean	Variance
a	16	42.31	33.20
b	15	44.00	11.13
c	16	43.12	8.66
d	14	40.36	8.74
e	16	40.56	11.60
f	16	44.13	15.80
g	14	40.07	16.25
h	16	44.00	24.80
i	16	44.00	8.00
j	13	43.31	5.90
k	14	41.88	15.55
l	16	43.12	12.90
m	16	44.69	7.54
n	13	41.07	8.90
o	16	42.31	3.71
p	15	41.73	8.06

The squares were arranged on the ground as indicated schematically in the following pattern:

```
   a b c
   d e f
   g h i
 j k l m
 n o p
```

We then obtain the following parameters for the distributions of the means and the variances

$$\begin{cases} \text{mean of variances } E(m) & = 42.54 \\ \text{variance of means } D^2(m) & = 2.18 \\ \text{mean of variances } E(\sigma^2) & = 13.19 \\ \text{variance of variances } D^2(\sigma^2) & = 56.90 \end{cases}$$

The value of the variance of the means, as was to be expected, is significantly greater than for random sub-populations. The squares have, in fact, really different assay values.

The sum of $D^2(m)+E(\sigma^2) = 15.37$ is, as it should, very close to the variance 15.30 of the boreholes in the whole deposit. The mean value of the variance of 13.19 is close to the value which one could expect. The mean value, in a 200×200 square, of the function $\sigma_d^2 = 7.9+5.76 \log d$ must be equal to twice the variance of the boreholes in this square. We find

$$\tfrac{1}{2}E(7.9+5.76 \log d) = 14.3$$

i.e. a theoretical value in fairly good agreement with the experimental value of 13.9 (less than 10 per cent deviation).

Finally, the variance—56.90—is considerably higher than that which we would have for random sub-populations $2/15(\sigma^4) = 23.1$.

The reason is the same as in the preceding paragraph. The boreholes in a square, drilled at rigid spacing in this square, are in negative correlation between themselves (hence an increase in the variance of the variances) of the same order of magnitude as in the preceding paragraph.

CONCLUSION

The logarithmic formula of the covariance is well confirmed for Mehengui. The small observed deviations which are of little significance from the strictly statistical point of view nevertheless undoubtedly correspond to real phenomena, but have a low enough amplitude to enable the formula to be used in spite of it. Instead of the covariance it is more convenient to study the variance σ_d^2 of the differences of the assay values of boreholes a distance of d apart, and to construct the curve for $\sigma_d^2 = f(d)$. The variance of sampling of a deposit-explored by n boreholes with spacing d is then given by the theoretical formula

$$\sigma_E^2 = \frac{1}{2n} f\left(\frac{d}{2.91}\right)$$

For a spacing of 200*m* as at Mehengui, this formula gives a variance of sampling of 0.62, in remarkable agreement with the experimental value which is 0.596.