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Change of Support for Diffusion-Type Random Functions[1]

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Diffusion-type random functions are a first attempt at a non-Markovian and multidimensional generalization of the Ito stochastic integral theory. Within this framework, the variation δf of the pdf can be evaluated for a small change of support. These results are compared with the provisions of the usual approximate models. The conclusions are: the affine correction is false for the first order approximation, unless there is a linear factor. The isofactorial model is true for the first order, and in the multi-Gaussian case, almost correct for the second-order approximation. Counterexamples are given for the discontinuous case, and a more general model is suggested.

KEY WORDS: diffusion processes, Ito conditions, conditional drift, conditional variogram, change of support.

INTRODUCTION

Change of support is probably the most important problem of Geostatistics today. In some parts of the mining industry, the support v of the selection is chosen by taking into account the information available at the time of selection (e.g., following the ore/waste contours). In this case, geostatisticians say that the geometry is *adaptative*, and only empirical approaches are possible. In what follows, we consider only the case of a *fixed geometry*, where the choice of the volume v is not influenced by any knowledge concerning the regionalization $z(x)$. In this case, exact mathematical results can be obtained, although the problem of the change of support remains extremely difficult.

Fundamentally, this problem is of a physical, not statistical nature, even if the language used here is probabilistic. Naturally, estimation problems also arise, but the influence of the support effect upon the recoverable reserves is generally much more important than any estimation error. The basic problem is to predict

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how the distributions, conditional or otherwise, assumed to be known for a point or quasi-point support, will be altered under a change of support. This physical law that we must infer involves the structure of the multivariate distributions in an essential way. In this context, speaking of distribution-free geostatistics sounds as unrealistic as, say, model-free physics.

In the first section, an introductory example of a nonprobabilistic nature is given. It suggests that the results of the following sections are also of a purely physical, not probabilistic, nature. Unfortunately, the method used in this first example is valid only for a differentiable function. For this reason, we use a probabilistic language in Sections 2 and 3 because it is the most convenient way of handling a continuous function without derivatives. But it must be kept in mind that the matter is physics, not probabilities. The definition of the diffusion-type random functions, and the expression of the variation δf of the pdf, are given in Section 2, and examples are presented in Section 3. In Sections 4 and 5, the predictions from the usual approximate models are compared with true distributions. In the discontinuous case, in particular when there is a strong nugget effect, the results of the present theory no longer hold, but a more general model, also applicable in the discontinuous case, is suggested.

1. AN INTRODUCTORY EXAMPLE

This first example is purely deterministic, not probabilistic. Its aim is to emphasize the physical nature of the change of support. For simplicity, we restrict ourselves to the one-dimensional case $x \in \mathbb{R}$, but the multidimensional case can be handled in the same way as is done in Section 3.3, but in a probabilistic framework.

We consider a function, or regionalized variable $z(x)$, $x \in \mathbb{R}$, which has *continuous derivatives* z', z'', and z'''. To eliminate any edge effect, this function $z(x)$ is assumed to be *periodic* with period L. Under these hypotheses, we may write

$$z(x+h) - z(x) = hz'(x) + (h^2/2)\, z''(x) + (h^3/6)\, R(x+h) \tag{1}$$

where the function R is *bounded* by a constant C independent of h.

Now, if the point x is replaced by a random variable X uniformly distributed on $(0, L)$, then $z(X)$, $z'(X)$, . . . become random variables. For simplicity, we assume that the distribution F of $z(X)$ has a pdf $f(z)$. This pdf $f(z)$ is defined by the relation

$$\int \varphi(z)\, f(z)\, dz = \frac{1}{L} \int_o^L \varphi[z(x)]\, dx \tag{2}$$

which holds for any function φ such that the integral is convergent.

The law of the change of support depends on the behavior of the function z in the neighborhood of the random point X, and so we must examine the variables $z(X+h)$ with $|h| \leqslant \epsilon$ for a given $\epsilon > 0$. Our tools will be conditional drift and the conditional variogram.

1. The Infinitesimal Conditional Drift

From relation (1), conditional drift is of the form

$$E[z(X+h) - z(X) | z(X) = z] = b(z; h) + o(\epsilon^2)$$

where the function $b(z; h)$ is defined by the relation

$$\int \varphi(z)\, b(z; h)\, f(z)\, dz = \frac{1}{L} \int_o^L \left[hz'(x) + \frac{h^2}{2} z''(x) \right] \varphi[z(x)]\, dx$$

which holds for any regular enough function φ.

But we have

$$\int_o^L \varphi[z(x)]\, z'(x)\, dx = \int_{z(o)}^{z(L)} \varphi(z)\, dz = 0$$

because the function $z(x)$ is periodic. In the same way, we find

$$\int_o^L z''(x)\, \varphi[z(x)]\, dx = -\int_o^L [z'(x)]^2\, \varphi'[z(x)]\, dx$$

Thus, the *infinitesimal conditional drift* $b(z; h)$ is of the form

$$\begin{cases} b(z; h) = h^2\, b(z) \\ b(z) = \frac{1}{2} E[z''(X) | z(X) = z] \end{cases} \tag{3}$$

and the function $b(z)$ is defined by

$$\int \varphi(z)\, b(z)\, f(z)\, dz = -\frac{1}{2L} \int_o^L [z'(x)]^2\, \varphi'[z(x)]\, dx \tag{4}$$

1.2. The Infinitesimal Conditional Variogram

In the same way, for $|h|$, $|h'| \leqslant \epsilon$, the conditional covariance is of the form

$$E\{[z(X+h) - z(X)]\,[z(X+h') - z(X)] | z(X) = z\} = C(z; h, h') + o(\epsilon^2)$$

and the infinitesimal conditional covariance $C(z; h, h')$ is

$$\begin{cases} C(z; h, h') = 2hh'a(z) \\ a(z) = \frac{1}{2} E[z'^2(X) | z(X) = z] \end{cases} \tag{5}$$

with the function $a(z)$ defined by

$$\int \varphi(z)\, a(z)\, f(z)\, dz = \frac{1}{2L} \int_o^L [z'(x)]^2\, \varphi[z(x)]\, dx \tag{6}$$

If $h = h'$, we find

$$\tfrac{1}{2} E\{[z(X+h) - z(X)]^2 | z(X) = z\} = h^2 a(z) + o(\epsilon^2)$$

so that the *infinitesimal conditional variogram* is

$$\gamma(z; h) = h^2 a(z)$$

The first relation (1-5) may be rewritten in the form

$$C(z; h, h') = \gamma(z; h) + \gamma(z; h') - \gamma(z; h - h') \tag{7}$$

This means that our conditional process behaves like an intrinsic random function in the neighborhood of the conditioning point.

1.3. A Differential Equation for the pdf

Now, by comparing (4) and (6), we find

$$\int \varphi b f\, dz = -\int \varphi' a f\, dz$$

Integrating by parts leads to

$$\int \varphi b f\, dz = \int \varphi \frac{d}{dz}(af)\, dz$$

for any function φ. Thus, the functions a and b and the pdf $f(z)$ satisfy the following differential equation

$$bf = (d/dz)(af) \tag{8}$$

In other words, the pdf is determined once the infinitesimal conditional drift and variogram are known, and we have

$$f(z) = \frac{C}{a(z)} \exp\left[\int^z \frac{b(u)}{a(u)}\, du\right]$$

where C is a suitable normalizing constant.

1.4. Moments of Order > 2

The moments of order > 2 may be neglected, because it follows from (1) that we have for any $|h_i| \leq \epsilon, i = 1, 2, \ldots, k > 2$

$$E\left[\prod_{i=1}^{k>2} |z(X_i + h_i) - z(X)| z(X) = z\right] = o(\epsilon^2) \tag{9}$$

1.5. The Law for a Small Change of Support

From these results, it is possible to find the expression for the variation δf of the pdf under a small change of support. Let $l = \epsilon$ be an infinitely small length, and consider the regionalized variable z_l defined by

$$z_l(x) = \frac{1}{l} \int_{-l/2}^{l/2} z(x+h)\, dh$$

From (1), we may write

$$z_l(x) = z(x) + (l^2/24)\, z''(x) + o(l^2)$$

Then, for any function φ which is bounded and has bounded derivatives φ' and φ'', for instance a complex exponential, we have

$$\varphi[z_l(x)] = \varphi[z(x)] + (l^2/24)\, z''(x)\, \varphi'[z(x)] + o(l^2)$$

If f_l is the pdf of the variable $z_l(X)$, and $\delta f = f_l - f$ is the variation of the pdf

$$\int \varphi(z)\, \delta f(z)\, dz = \frac{l^2}{24} \frac{1}{L} \int_o^L z''(x)\, \varphi'[z(x)]\, dx$$

$$= -\frac{l^2}{24} \frac{1}{L} \int_o^L [z'(x)]^2\, \varphi''[z(x)]\, dx$$

By comparing with (6) and integrating by parts, we conclude

$$\int \varphi(z)\, \delta f(z)\, dz = -\frac{l^2}{12} \int \varphi''(z)\, a(z)\, f(z)\, dz$$

$$= -\frac{l^2}{12} \int \varphi(z) \frac{d^2}{dz^2}(af)\, dz$$

and finally, by also taking into account the differential equation (8)

$$\delta f = -(l^2/12)(d^2/dz^2)(af) = -(l^2/12)(d/dz)(bf) \tag{10}$$

In geostatistics, the *tonnage* $T(z)$ selected above a given cutoff grade z, and the corresponding conventional *benefit* function $B(z)$ (Matheron, 1983a), are defined by

$$\begin{cases} T(z) = 1 - F(z) = \displaystyle\int_z^\infty f(u)\, du \\ B(z) = \displaystyle\int_z^\infty (u - z)\, f(u)\, du = \int_z^\infty T(u)\, du \end{cases}$$

By (10), the variations of these functions under a small change of support are

$$\delta T = (l^2/12)\, bf \qquad \delta B = -(l^2/12)\, af \tag{11}$$

In other words, under a small change of support and for a given cutoff grade z, the variation of the tonnage is proportional to the conditional drift, and *the loss* (in dollars!) *is proportional to the conditional variogram.*

Another important geostatistical function is $Q(T)$ which represents the quantity of recoverable metal for a given selected tonnage T, defined by

$$Q(T) = B[z(T)] + Tz(T)$$

where $z(T)$ is the inverse function of $T(z)$. Thus, for *fixed* T, the variation of $Q(T)$ is

$$\delta Q(T) = \delta B(z) + \delta z(T)(\partial B/\partial z) + T\delta z(T)$$

But we have $\partial B/\partial z = -T$, and finally

$$\delta Q(T) = \delta B(z) = -(l^2/12)\, af \qquad [\text{with } z = z(T)] \tag{12}$$

Under a small change of support and for a given selected tonnage T, *the loss in metal is also proportional to the conditional variogram.*

The function $z(T)$ is the classical T quantile. For a fixed T, its variation is given by the condition

$$\delta T(z) + \delta z \frac{\partial T}{\partial z} = 0$$

But $\partial T/\partial z = -f(z)$. Hence $\delta z(T) = \delta T/f$ and, by (11)

$$\delta z(T) = (l^2/12)\, b(z)$$

2. DIFFUSION-TYPE RANDOM FUNCTIONS

These fundamental results, i.e., the differential equation (8) and the expression (10) for the law for a small change of support, were obtained under strong regularity hypotheses. We now generalize them inside a much wider framework, but this requires the use of probabilistic language. In a sense, our goal is a multi-dimensional and non-Markovian generalization of the Ito stochastic integral (Guikhman and Skorokhod, 1972). Roughly speaking, it is assumed that, in the neighborhood of a conditioning point, the conditional random function behaves like a Gaussian intrinsic random function.

The physical implications of this model are emphasized by the terminology: diffusion-type random function. It may be said that this has implicitly worked as an heuristic model since the beginning, so that it might be responsible for the successes of Geostatistics up to now.

2.1. The Ito Conditions

We consider a stationary, but in general not multi-Gaussian, random function $Z(x)$, $x \in \mathbb{R}^n$, and we want to study the behavior of the conditional version of $Z(x)$ in the neighborhood of a given conditioning point x_o. Without loss of generality, this point x_o may be chosen as the origin of the coordinates, i.e., $x_o = 0$. More precisely, if v *is a small neighborhood* of $x_o = 0$ and $\gamma(h)$ is the variogram of the R.F., $Z(x)$, we put

$$\epsilon = \sup \{\gamma(x-y) \quad x, y \in v\}$$

and the notation $o(\epsilon)$ will denote a quantity such that $o(\epsilon)/\epsilon \to 0$ if $\epsilon \to 0$. In particular, the present theory holds only if there is no nugget effect.

The following conditions, which will be called *Ito conditions*, even though they are not in quite the standard form, are assumed to be satisfied for any x, $y, x_i \in v$

$$E[Z(x) - Z_o | Z_o = z] = b(z; x) + R_1$$
$$E[(Z(x) - Z_o)(Z(y) - Z_o) | Z_o = z] = C(z; x, y) + R_2 \tag{13}$$
$$E\left[\prod_{i=1}^{k>2} |Z(x_i) - Z_o| \, | Z_o = z\right] = R_3$$

with remainders R_1, R_2, R_3 satisfying bounds of the form

$$|R| \leqslant o(\epsilon) H \quad E[H^n] < \infty, n = 1, 2, \ldots \tag{14}$$

We also assume that the random variable $b(Z_o; x)$ and $C(Z_o; x, y)$ have finite moments for all the required orders.

The function $b(z; x)$ is the *infinitesimal conditional drift*; $C(z; x, y)$ is the *infinitesimal conditional covariance*, and the *infinitesimal conditional variogram* $\gamma(z; h)$ is defined by

$$\gamma(z; h) = \tfrac{1}{2} C(z; h, h) \qquad (h \in v)$$

2.2. A Differential Equation for the pdf

Let φ be a bounded function with bounded derivatives φ', φ'' and φ'''. For instance, φ may be a complex exponential. Then, for any $x \in v$, we may write

$$\varphi(Z_x) - \varphi(Z_o) = (Z_x - Z_o)\varphi'(Z_o) + \tfrac{1}{2}(Z_x - Z_o)^2 \varphi''(Z_o) + [(Z_x - Z_o)^3/6]\, U$$

with a random variable U such that $|U| \leqslant \sup |\varphi'''(z)| < \infty$. It follows from the Ito conditions

$$E[\varphi(Z_x) - \varphi(Z_o)/Z_o = z] = b(z; x)\varphi'(z) + \gamma(z, x)\varphi''(z) + R$$

with a remainder R satisfying a bound of the form (14). Taking the expectation

$$E[\varphi(Z_x)] - E[\varphi(Z_o)] = E[b(Z; x)\varphi'(Z) + \gamma(Z; x)\varphi''(Z)] + o(\epsilon)$$

But this expression is zero because of stationarity. Thus we have

$$\int b(z;x)\,\varphi'(z)\,f(z)\,dz = -\int \gamma(z;x)\,\varphi''(z)\,f(z)\,dz + o(\epsilon)$$

$$= \int \varphi'(z)\,(\partial/\partial z)[\gamma(z;x)\,f(z)]\,dz + o(\epsilon)$$

The term $o(\epsilon)$ may be dropped out, because the functions b and γ are defined up to $o(\epsilon)$-type quantities only. This relation holds for any function φ, and so

$$b(z;x)\,f(z) = (\partial/\partial z)[\gamma(z;x)\,f(z)] \tag{15}$$

Thus, the pdf is determined once the infinitesimal conditional drift and variogram are given. From another point of view, relation (15) also shows that the two functions b and γ do not behave independently. An equivalent form, which is useful, is the following

$$E[b(Z;x)\,\varphi'(Z) + \gamma(Z;x)\,\varphi''(Z)] = 0 \tag{16}$$

2.3. Structure of the Conditional Covariance

Let x and y be two points in v. Consider the random variables

$$Z_\alpha = \alpha Z(x) + (1-\alpha)\,Z(y) = Z_o + \alpha(Z_x - Z_o) + (1-\alpha)(Z_y - Z_o)$$

Then, for any bounded function φ with bounded derivatives φ', φ'', and φ''', we have

$$\varphi(Z_\alpha) = \varphi(Z_o) + (Z_\alpha - Z_o)\,\varphi'(Z_o) + \tfrac{1}{2}(Z_\alpha - Z_o)^2\,\varphi''(Z_o) + \tfrac{1}{6}(Z_\alpha - Z_o)^3 U$$

with $|U| \leqslant \sup|\varphi'''| < \infty$. By the Ito conditions, it follows that

$$E[\varphi(Z_\alpha)|Z_o = z] = \varphi(z) + [\alpha b(z;x) + (1-\alpha)\,b(z;y)]\,\varphi'(z) + [\alpha^2\,\gamma(z;x) + (1-\alpha)^2\,\gamma(z;y) + \alpha(1-\alpha)\,C(z;x,y)]\,\varphi''(z) + R$$

and the remainder R satisfies a bound of the form (14). Thus, by taking the expectation, we have

$$\begin{aligned}E[\varphi(Z_\alpha)] = {}& E[\varphi(Z_o)] + E[b(Z,y)\,\varphi'(Z) + \gamma(Z;y)\,\varphi''(Z)]\\ &+ \alpha E\{[b(Z,x) - b(Z,y)]\,\varphi'(Z) + [C(Z,x,y) - 2\gamma(Z,y)]\,\varphi''(Z)\}\\ &+ \alpha^2 E\{[\gamma(Z,x) + \gamma(Z,y) - C(Z;x,y)]\,\varphi''(Z)\} + o(\epsilon)\end{aligned}$$

From (16), this may be rewritten

$$E[\varphi(Z_\alpha)] - E[\varphi(Z_o)] = \alpha(1-\alpha)\,E\{[C(Z;x,y) - \gamma(Z,x) - \gamma(Z,y)]\,\varphi''(Z)\}$$

Now, because of stationarity, this expression does not change when x and y

are replaced by $x+h$ and $y+h$. In particular, if $h = -y$, we have $\gamma(Z;o) = C(Z;x-y,o) = 0$, and find

$$E\{[C(Z;x,y) - \gamma(Z,x) - \gamma(Z,y)]\,\varphi''(Z)\} = -E[\gamma(Z;x-y)\,\varphi''(Z)]$$

for any regular enough function φ, and thus

$$C(z;x,y) = \gamma(z,x) + \gamma(z,y) - \gamma(z;x-y) \tag{17}$$

This relation means that, in the neighborhood of the conditioning point $x_o = 0$, the conditional random function behaves like an intrinsic random function, except for the drift $b(z;x)$.

2.4. Variation of the pdf Under a Small Change of Support

Now consider the variable $Z_v = (1/v)\int_v Z(x)\,dx$, or more generally, the variable

$$Z_\mu = \int Z(x)\,\mu(dx)$$

where μ is a measure with its support $\subset v$ and such that $\int \mu(dx) = 1$. Then, for any regular enough function φ we find as above

$$\varphi(Z_\mu) - \varphi(Z_o) = (Z_\mu - Z_o)\,\varphi'(Z_o) + \tfrac{1}{2}(Z_\mu - Z_o)^2\,\varphi''(Z_o) + \tfrac{1}{6}(Z_\mu - Z_o)^3 U$$

$$|U| \leqslant \sup|\varphi'''| < \infty$$

and, by the Ito conditions

$$E[\varphi(Z_\mu) - \varphi(Z_o)] = E\left[\varphi'(Z)\int b(Z,x)\,\mu(dx) + \tfrac{1}{2}\varphi''(Z)\iint C(Z;x,y)\,\mu(dx)\,\mu(dy)\right] + o(\epsilon)$$

But, from (17), we have

$$\tfrac{1}{2}\iint C(z;x,y)\,\mu(dx)\,\mu(dy) = \int \gamma(z,x)\,\mu(dx) - \tfrac{1}{2}\bar{\gamma}(z)$$

where $\bar{\gamma}(z)$ is defined by

$$\bar{\gamma}(z) = \iint \gamma(z;x-y)\,\mu(dx)\,\mu(dy)$$

and from (16)

$$E\left[\varphi'(Z)\int b(Z;x)\,\mu(dx) + \varphi''(Z)\int \gamma(z;x)\,\mu(dx)\right] = 0$$

and so we find

$$E[\varphi(Z_\mu) - \varphi(Z_o)] = -\tfrac{1}{2}E[\varphi''(Z)\,\bar{\gamma}(Z)] + o(\epsilon) \tag{18}$$

Thus, the *variation* $\delta f = f_\mu - f$ of the pdf satisfies the following relations

$$\int \varphi(z)\,\delta f(z)\,dz = -\tfrac{1}{2}\int \varphi''\bar{\gamma}(z)\,f(z)\,dz = -\tfrac{1}{2}\int \varphi\,\frac{d^2}{dz^2}(\bar{\gamma}f)\,dz$$

for any regular enough function. It follows that

$$\delta f(z) = -\tfrac{1}{2}(d^2/dz^2)[\bar{\gamma}(z)\,f(z)] \tag{19}$$

From the differential eq. (15), we find that the variation of the *tonnage* $T(z) = 1 - F(z)$ is given by

$$\delta T(z) = \tfrac{1}{2}\bar{b}(z)\,f(z)$$

where $\bar{b}(z) = \iint b(z; x-y)\,\mu(dx)\,\mu(dy)$ is the average of the conditional drift. Concerning the function $B(z)$ and $Q(T)$, we find exactly as in Section 1

$$\delta B(z) = \delta Q(T) = -\tfrac{1}{2}\bar{\gamma}(z)\,f(z) \tag{20}$$

In the same way, the variation of the *quantile* $z(T)$ is

$$\delta z(T) = \tfrac{1}{2}\bar{b}(z)$$

N.B. From the relation $\int B(z)\,dz = \frac{1}{2}E(Z^2)$ (Matheron, 1983a), we always have

$$\int \delta B(z)\,dz = \tfrac{1}{2}\delta\sigma^2 = -\tfrac{1}{2}\bar{\gamma} = -\tfrac{1}{2}\int \bar{\gamma}(z)\,f(z)\,dz$$

The relation (20) shows that this general rule also holds in the conditional sense, in the case of a diffusion-type random function. It means that, for a given cutoff grade z, the loss due to a small change of support v is proportional to the average value in v of the conditional variogram.

2.5. The Case of the Proportional Effect

In the general case, the behavior of the conditional variogram $\gamma(z; h)$, and in particular its kind of anisotropy, may depend in a complex way on the conditioning value z. The simplest possible model is the case of the proportional effect, i.e.

$$\gamma(z; h) = a(z)\,\gamma(h)$$

where $\gamma(h)$ is the nonconditional variogram. In this case, we also find by (15)

$$b(z; h) = b(z)\,\gamma(h)$$

and the functions a, b, and f satisfy the differential equation

$$(d/dz)(af) - bf = 0 \tag{21}$$

In particular, the pdf is of the form

$$f(z) = \frac{C}{a(z)}\exp\left[\int^z \frac{b(u)}{a(u)}\,du\right]$$

Consequently, the variations of the functions $f(z)$, $T(z)$, $B(z)$, and $Q(T)$ under a small change of support, are

$$\begin{cases} \delta f(z) = -(\bar{\gamma}/2)(d/dz)(bf) = -(\bar{\gamma}/2)[(ab' - ba' + b^2)/a]f \\ \delta T(z) = (\bar{\gamma}/2)\,bf \\ \delta B(z) = \delta Q(T) = -(\bar{\gamma}/2)\,af \end{cases}$$

If $u(z)$ is a regular enough function, we have

$$\delta E[u(Z)] = (\bar{\gamma}/2)\,E[b(Z)\,u'(Z)] = -(\bar{\gamma}/2)\,E[a(Z)\,u''(Z)] \tag{22}$$

and in particular the variation of the moment of order n is

$$\delta M_n = (n/2)\,\bar{\gamma}E[Z^{n-1}b(Z)] = -[n(n-1)/2]\,\bar{\gamma}E[Z^{n-2}a(Z)] \tag{23}$$

Note that we have

$$E[a(Z)] = 1 \tag{24}$$

because, by definition, $\gamma(z; h) = a(z)\,\gamma(h)$ and $\gamma(h) = E[\gamma(Z; h)]$. Thus, in the particular case $n = 2$, (23) becomes

$$\delta M_2 = \delta\sigma^2 = -\bar{\gamma}$$

as expected.

This very simple model of the proportional effect will probably prove to be very useful in application. But it must be kept in mind that more complex models are also possible.

3. EXAMPLES

It is not obvious a priori that stationary diffusion-type random functions in $\mathbb{R}^n$ do exist. I give three examples, but I hope that many others do exist.

3.1. Diffusion Markov Processes on the Straight Line

This first example is obvious, because it is the starting point of the whole theory. It is the stationary solution to the Ito stochastic differential equation

$$dZ_t = b(Z_t)\,dt + [a(Z_t)]^{1/2}\,d\xi_t$$

where ξ_t is a standard brownian motion, with $E(\xi_t) = 0$ and $\mathrm{Var}(\xi_t) = 2t$. The Ito conditions are satisfied, and we are in the case of the proportional effect (see Section 2.5). The only required condition is the existence of the stationary pdf

$f(z)$, i.e.

$$\int \frac{1}{a(z)} \exp\left(\int_o^z \frac{a(u)}{b(u)}\,du\right) dz < \infty$$

The *infinitesimal generator* A of the Markov process Z_t is the operator

$$A = a(z)(d^2/dz^2) + b(z)(d/dz) \tag{25}$$

For any regular enough function φ, the function

$$\varphi_t(z) = E[\varphi(Z_t)|Z_o = z]$$

is the unique solution of the *evolution equation*

$$\frac{d\varphi_t}{dt} = A\varphi_t \qquad (\varphi_o = \varphi)$$

so that we can write

$$\varphi_t = P_t\varphi = e^{At}\varphi$$

or explicitly

$$\varphi_t(z) = \int P_t(z; dz')\,\varphi(z')$$

where $P_t(z; dz')$ is the transition probability of the Markov process Z_t. This means that the stationary bivariate distribution

$$f_t(z_o, z')\,dz_o\,dz' = f(z_o)\,dz_o\,P_t(z_o; dz')$$

of (Z_o, Z_t) is determined once the operator A, i.e., the two functions $a(z)$ and $b(z)$, is known.

Now, if $Z(x)$ is a diffusion-type random function in $\mathbb{R}^n$ with a proportional effect, we may also consider the Markov process Z_t on $\mathbb{R}$ defined by the operator (25), with the same functions a and b. The pdf $f(z)$ is the same for $Z(x)$ and Z_t. But what happens to the bivariate distributions? Are they of the same form or, more precisely, for any $h = x - y \in \mathbb{R}^n$, is it possible to find a value of t such that

$$f_h(z(x), z(y)) = f_t(z_o, z_t)? \tag{26}$$

The answer to this question is very important for practical application, because any consistent nonlinear estimation procedure requires the knowledge of at least the bivariate distributions $(Z(x), Z(y))$ and $(Z(x), Z(v))$.

The answer is in the affirmative in the multi-Gaussian case, which is our second example. It is doubtful in the case of the third example, and probably negative in the general case. For this reason, I suggest a "call for research" of the form: Under what condition is the answer to the question (26) positive?

3.2. The Multi-Gaussian Case

Let $Y(x)$ be a stationary multi-Gaussian random function. Without loss of generality, we may assume

$$E[Y(x)] = 0$$

$$\mathrm{Var}[Y(x)] = 1$$

$$\mathrm{Cov}(Y_x, Y_y) = \rho(x-y) = 1 - \gamma_o(x-y)$$

We assume that there is no nugget effect, i.e., the variogram γ_o is continuous. Then the Ito conditions are satisfied with a proportional effect, and we have

$$b_o(y; x) = -y\gamma_o(x) \qquad \gamma_o(y; h) = \gamma_o(h)$$

Now we may define a new random function by putting

$$Z(x) = \varphi[Y(x)]$$

Then, if the transform φ is regular enough, $Z(x)$ is also a diffusion-type random function with a proportional effect. If we assume for instance that φ, φ', φ'', and φ''' satisfy an exponential bound of the form

$$|\varphi| + |\varphi'| + |\varphi''| + |\varphi'''| \leqslant B\exp(C|y|)$$

it is easy to show that the Ito conditions are satisfied and that we have a proportional effect

$$b(z; x) = b(z)\,\gamma_o(h)$$

$$\gamma(z; h) = a(z)\,\gamma_o(h)$$

Note that these functions a and b are not exactly the same as in Section 2.5, because the variogram $\gamma(h)$ of $Z(x)$ is proportional but not equal to $\gamma_o(h)$. If A_o is the generator of the standard Gaussian Markovian process Y_t, i.e.

$$A_o = (d^2/dy^2) - y(d/dy)$$

it is easy to see that the functions $a(z)$ and $b(z)$ are given by

$$\begin{aligned} b(z) &= E[A_o\varphi(Y)|\varphi(y) = z] \\ a(z) &= E[\varphi'^2(Y)|\varphi(Y) = z] \end{aligned} \tag{27}$$

The appearance of the conditional expectation is due to the fact that generally the equation $\varphi(y) = z$ may have more than one solution. If the transform φ is a true anamorphosis, i.e., is strictly increasing, the conditional expectation can be dropped out, and we have

$$\begin{aligned} b(z) &= A_o\varphi(y) \\ a(z) &= \varphi'^2(y) \end{aligned} \tag{28}$$

with y uniquely determined by $z = \varphi(y)$.

Note that transformations (27) or (28) are general; if $Y(x)$ is a diffusion type R.F. with a proportional effect defined by two functions $a_o(y)$ and $b_o(y)$, and if the transform φ is regular enough, the R.F. $Z(x) = \varphi[Y(x)]$ is also of the diffusion type with a proportional effect, and the new functions $a(z)$, $b(z)$ are given by

$$b(z) = E[A_o \varphi(Y) | \varphi(Y) = z]$$

$$a(z) = E[a_o(Y)\, \varphi'^2(Y) | \varphi(Y) = z]$$

where

$$A_o = a_o(y)(d^2/dy^2) + b_o(y)(d/dy)$$

3.3. The Regular Case

Now, assume the stationary R.F. $Z(x)$ to be three times differentiable in quadratic mean. More precisely

$$Z(x) = Z_o + x^i \partial_i Z_o + \tfrac{1}{2} x^i x^j \partial_{ij} Z_o + o(|x|)\, R$$

with $|R| \leqslant H$ for a random variable H independent of the point x and such that $E[H^n] < \infty$ for any n. We also assume that the partial derivatives $\partial_i Z_o$, $\partial_{ij} Z_o$, and Z_o itself have finite moments of all required orders.

Then it is easy to show that the Ito conditions are satisfied with

$$b(z; x) = x^i x^j b_{ij}(z)$$

$$b_{ij}(z) = \tfrac{1}{2} E[\partial_{ij} Z_o | Z_o = z]$$

$$\gamma(z; h) = h^i h^j a_{ij}(z)$$

$$a_{ij}(z) = \tfrac{1}{2} E[\partial_i Z_o \partial_j Z_o | Z_o = z]$$

In the case of a proportional effect, there is a constant positive definite matrix K_{ij} such that

$$a_{ij}(z) = a(z)\, K_{ij}$$

$$b_{ij}(z) = b(z)\, K_{ij}$$

But there is no reason for these relations to be satisfied in the general case.

4. APPROXIMATE MODELS OF CHANGE OF SUPPORT

The above results, and in particular formula (19), can be used only in the case of a very small change of support. But in practice the variance reduction is often greater than 50%. In the case of a relatively large change of support, two kinds of approximate models are available:

- the affine correction (Journel and Huijbregts, 1978)
- the isofactorial models (Matheron, 1975a, 1983b)

Especially popular is the very particular case called discretized Gaussian model (Matheron, 1975b; Journel and Huijbregts, 1978). A more general presentation is given in Matheron (1983b) in the discrete case.

In this section, we compare the predictions of these models with the true distributions, in the case of a diffusion-type random function with proportional effect. The comparison will be made for the first-order approximation in the general case, and for the second-order in the multi-Gaussian case. Moreover, we give actual values in a few particular examples. The general conclusions will be as follows

- *Isofactorial model predictions are exact for the first-order approximation and, in the multi-Gaussian case, are alsmost correct for the second-order approximation.*
- *Affine correction is false* for the first-order onward, *unless* z - m *is a factor*, i.e., an eigenfunction for the operator $A = a(z)(d^2/dz^2) + b(z)(d/dz)$.

4.1. The Affine Correction (A.C.)

In this very simple model, it is assumed that $(Z_v - m)/\sigma_v$ and $(Z_o - m)/\sigma$ have the same distribution. Under a small change of support, the variation $\sigma_v^2 - \sigma^2 = \bar{\gamma}$ of the variance is small. Thus, by putting

$$\epsilon = 1 - (\sigma_v/\sigma) = (\bar{\gamma}/2\sigma^2) + o(\bar{\gamma}/\sigma^2)$$

Z_v and $Z_o - \epsilon(Z_o - m)$ have the same distribution. Thus, for any regular enough function $u(z)$, the A.C. prediction is

$$E[u(Z_v)] = E[u(Z_o)] - (\bar{\gamma}/2\sigma^2)\, E[(Z_o - m)\, u'(Z_o)] + o(\bar{\gamma})$$

By comparison with (22), we see that this expression is correct only if we have

$$E[b(Z)\, u'(Z)] = -(1/\sigma^2)\, E[(Z - m)\, u'(Z)]$$

for any function u, i.e.

$$b(z) = -(Z_o - m)/\sigma^2$$

In terms of the differential operator

$$A = a(z)(d^2/dz^2) + b(z)(d/dz)$$

associated with the diffusion type R.F. $Z(x)$, this condition may be written as

$$A(z - m) = -\lambda(z - m)$$

Hence the conclusion: the *affine correction is false for the first order, unless* (z - m) *is an eigenfunction* of the operator A.

In the *multi-Gaussian case* of Section 3.2, the conclusion is as follows:

If φ is a *true* anamorphosis, i.e., a strictly increasing transform, the affine correction is exact for the first order if and only if

$$A_o(\varphi - m) = -\lambda(\varphi - m)$$

i.e. if $\varphi - m$ is an *Hermite's polynomial*; but Hermite's polynomials of degree $n \geqslant 2$ are not monotonic, so that in fact the affine correction is false for any nonlinear anamorphosis.

Now, if φ is a regular transform, but not strictly monotonic, the affine correction is exact for the first order if $\varphi - m$ is an Hermite's polynomial, or more generally, if

$$E[A_o\varphi(Y) | \varphi(Y) = z] = -\lambda(z - m)$$

This condition is satisfied, for instance, if $\varphi(y) = y^2$, and also if $\varphi(y) = y^2 1_{y>o}$, with $1_{y>o} = 1$ if $y > 0$, and 0 otherwise.

4.2. The Isofactorial Model

The starting point of this model is the Cartier condition

$$E[Z(x)/Z_v] = Z_v$$

where the random point x is uniformly distributed inside v (Matheron, 1983a, b). Moreover, the bivariate distribution of $(Z(x), Z_v)$ is assumed to be the same as if we had for a suitable $t > 0$

$$Z_v = \varphi_v(Z_o) \qquad Z(x) = Z_t$$

where Z_t is the diffusion Markov process defined by the generator

$$A = a(z)(d^2/dz^2) + b(z)(d/dz)$$

From Cartier's condition, this comes back to putting

$$Z_v = \varphi_v(Z)$$

$$\varphi_v = P_t z = e^{At} z$$

so that, for the first-order approximation, we find

$$\varphi_v(z) = z + tAz + o(t) = z + tb(z) + o(t)$$

The parameter t must be chosen so that the variance has the correct value σ_v^2. For the first-order approximation, we have

$$E[\varphi_v^2(Z)] = E(Z^2) + 2tE(ZA\,Z) + o(t)$$

But, from the differential eq. (21), we find

$$E[u(Z)\,Av(Z)] = -E[a(Z)\,u'(Z)\,v'(Z)]$$

and thus $E(ZAZ) = -E[a(Z)] = -1$ from (24). It follows that the variance has the correct value if

$$t = \bar{\gamma}/2$$

and finally the model is

$$\varphi_v(z) = z + (\bar{\gamma}/2)\, b(z) + o(\bar{\gamma})$$

For any regular enough function u, it follows

$$E\{u[\varphi_v(Z)]\} = E[u(Z)] + (\bar{\gamma}/2)\, E[b(Z)\, u'(Z)] + o(\bar{\gamma})$$

By comparison with (22), this is the correct value, and we conclude

The isofactorial model is exact for the first-order approximation.

4.3. The Second-Order Approximation

In *the multi-Gaussian case* $Z(x) = \varphi[Y(x)]$ of Section 3.2, it is possible to obtain the second-order approximation formula for the variation δf of the pdf $f(z)$. After routine calculations, we find for any regular enough function $u(z)$

$$E[u(Z_v)] = E[u(Z_o)] - (\bar{\gamma}_o/2)\, E[u''(Z_o)\,\varphi'^2(Y_o)] + \frac{1}{2}\left[\frac{u''' \cdot (\varphi')^4}{4}(\bar{\gamma}_o)^2 + u''' \cdot (\varphi')^2\, \overline{\gamma_o\gamma_o'} + \frac{1}{2}u'' \cdot (\varphi'')^2\, \overline{\gamma_o^2}\right] + o(\bar{\gamma}^2)$$

In particular, if $u(z) = z^n$, the order n moment $M_n(v)$ is given by

$$M_n(v) = M_n(o) - [n(n-1)/2]\, \bar{\gamma}_o E[Z^{n-2}\varphi'^2(Y)] + \frac{1}{2}E\left[\frac{n(n-1)(n-2)(n-3)}{4}(\bar{\gamma}_o)^2 Z^{n-4}\varphi'^4 + n(n-1)(n-2)\,\overline{\gamma_o\gamma_o'}\, Z^{n-3}\varphi'^2 + \frac{n(n-1)}{2} Z^{n-2}\varphi''^2\,\overline{\gamma_o^2}\right] + o(\bar{\gamma}^2)$$

Note the occurrence of the term

$$\overline{\gamma_o\gamma_o'} = \frac{1}{v^3}\iiint \gamma_o(x-y)\,\gamma_o(x-y')\,dx\,dy\,dy'$$

For the second-order approximation, we find after some calculations that the prediction of the isofactorial model is correct, except this term $\overline{\gamma_o\gamma_o'}$ is replaced by $(\bar{\gamma}_o)^2$. From a numerical point of view, the difference is very small. For instance, in the one-dimensional case, if $\gamma_o(h) = |h| + o(|h|^2)$, we find

$$\overline{\gamma_o\gamma_o'} = \tfrac{7}{60}h^2 + o(h^2) = 0.11\,667h^2 + o(h^2)$$

$$(\bar{\gamma}_o)^2 = \tfrac{1}{9}h^2 + o(h^2) = 0.11\,111h^2 + o(h^2)$$

Hence the conclusion: in the multi-Gaussian case, the isofactorial model is almost correct for the second-order approximation.

4.4. The Lognormal Case

In this model, we have

$$Z(x) = me^{sY(x)-(s^2/2)}$$

where $Y(x)$ is a standardized stationary multi-Gaussian random function, as in Section 3.2. In this case, the prediction of the isofactorial model is another lognormal distribution with the same mean m, i.e.

$$Z_v = me^{s_v Y - s_v^2/2}$$

and the parameter s_v is determined so that the variance has the correct value σ_v^2 (Matheron, 1975b).

But in this case the true values of the order n moment $M_n(v)$ are

$$M_n(v) = E[Z_v^n] = \frac{1}{v^n} \int_{v^n} E[Z_{x_1} Z_{x_2}, \ldots, Z_{x_n}]\, dx_1\, dx_2, \ldots, dx_n$$

$$= \frac{1}{v^n} M_n(o) \int_{v^n} e^{-s^2 \Sigma_{i<j} \gamma_o(x_i - y_j)}\, dx_1\, dx_2, \ldots, dx_n$$

and so numerical calculations are always possible, at least for $n = 3$ and 4.

In the one-dimensional case, if the variogram of $Y(x)$ is

$$\gamma_o(h) = |h| \quad \text{if } |h| \leqslant 1$$
$$= 1 \quad \text{otherwise}$$

explicit formulas were obtained, after fairly tedious calculations. Figures 1-4 show the results in the case $m = s = 1$, for the variable

$$Z_v = \frac{1}{t} \int_o^t Z_x\, dx$$

Figure 1 shows the relative order 3 moment $M_3(t)/M_3(o)$. At this scale, true values and isofactorial predictions cannot be distinguished. It is surprising that the order two approximation formula is better than affine correction prediction.

Figure 2 shows that a much clearer discrimination is obtained by using the order 3 relative cumulant

$$\frac{\chi_3(t)}{\sigma^3(t)} = E\left[\left(\frac{Z_v - m}{\sigma_v}\right)^3\right]$$

The true values are decreasing because of the convergence toward normality. The isofactorial predictions are extremely good if $t \leqslant 1$. However, in the affine cor-

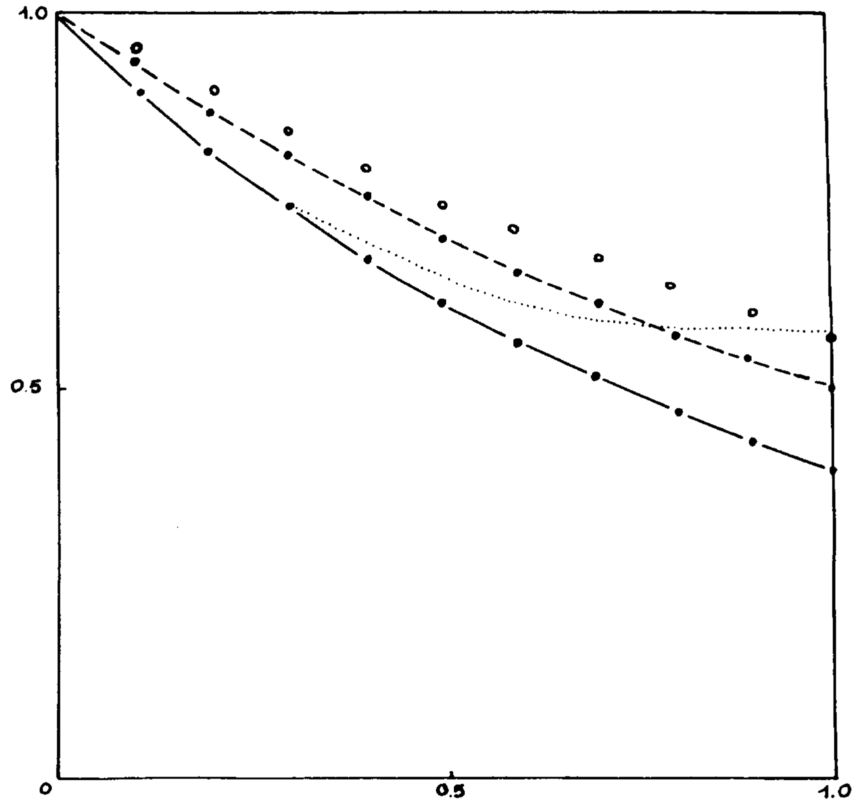

Fig. 1. Lognormal process 1 – D – $M_3(t)/M_3(o)$; $s = 1$; $\sigma^2/m^2 = 1.71828\ldots$; $\rho(h) = (1 - |h|)_+$. Long dash/dot line represents *true values* of $M_3(t)/M_3(o)$, and *isofactorial predictions*: they cannot be distinguished at this scale! Short dash dot line equals the A.C. prediction. Small dotted line represents the order two approximation; it is better than the A.C. prediction. Large dotted line equals the relative variance $\sigma^2(t)/\sigma^2(o)$.

rection model, χ_3/σ^3 remains constant because $(Z_v - m)/\sigma_v$ and $(Z - m)/\sigma$ have the same distribution.

Figure 3 shows the order 4 relative cumulant χ_4/σ^4: the conclusions are the same.

Figure 4 shows (on a logarithmic scale) the long run behavior of the relative cumulant χ_3/σ^3. From the general theory (convergence toward normality) we know that the true values are $\sim 1/t^{1/2}$ for large enough t. The isofactorial prediction is also $\sim 1/t^{1/2}$ (this is a general result). If t is greater than or equal to 5 or 10, the ratio isofactorial prediction/true value remains constant approximately equal to 0.78, while the affine correction prediction remains constant!

N.B. In the multi-Gaussian case, we saw that the isofactorial model is cor-

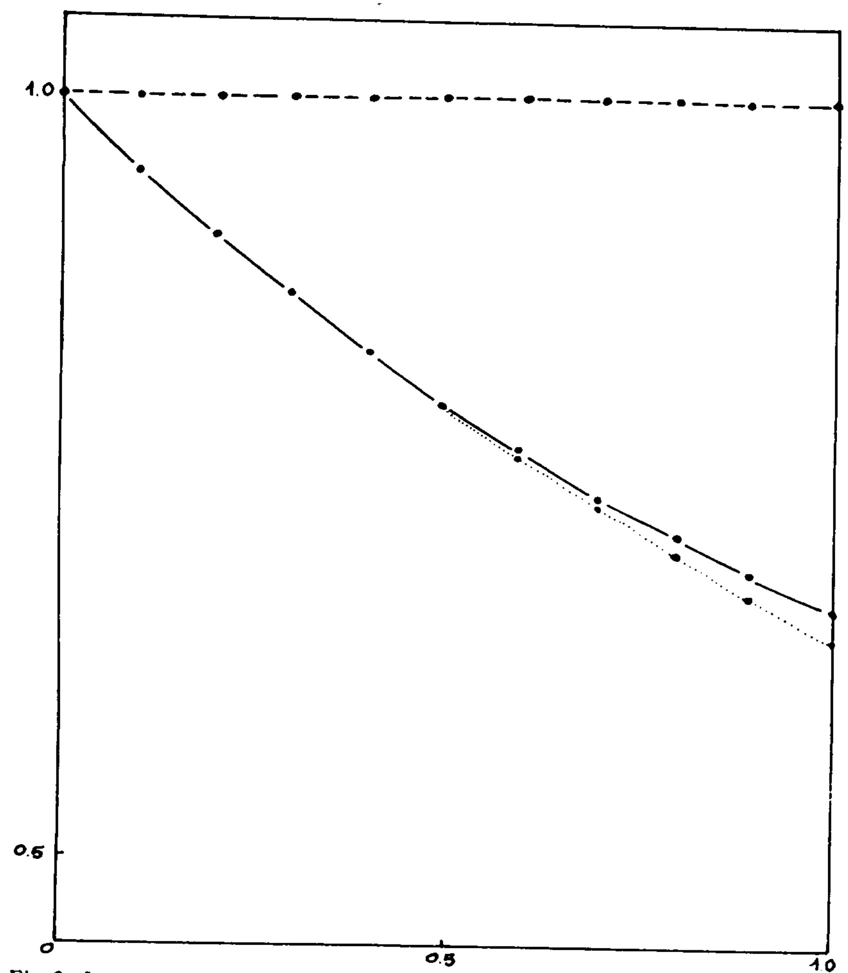

Fig. 2. Lognormal process--*order 3 relative cumulant* $\chi_3(t)$. Long dash/dot line represents true values. Dotted line equals isofactorial predictions. Short dash/dot line equals A.C. predictions (remains constant!).

rect for a small change of support. It turns out that it is also correct for a very large change of support, because the convergence of $[\varphi_v(Y) - m]/\sigma_v$ toward a normal variable $N(0, 1)$ automatically holds. This is the reason why the isofactorial model appears really good for any change of support in this case.

4.5. The Gamma Diffusion Process

If $Y(t)$ is the Markov process defined by the generator

$$A = a(y)(d^2/dy^2) - b(y)(d/dy)$$

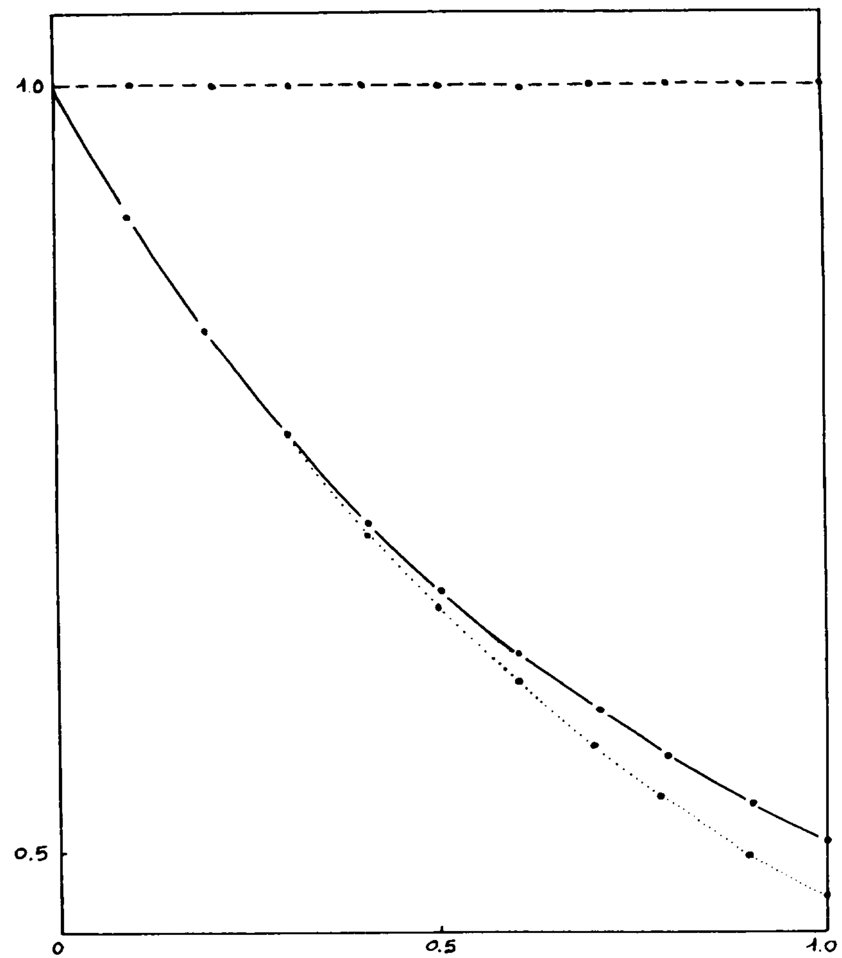

Fig. 3. Lognormal 1 - *D*--Order 4 relative cumulant $\chi_4(t)/\sigma^4(t)$. Legend as in Fig. 2.

put

$$H_t(\lambda; y) = E[e^{-\lambda \int_0^t \varphi(Y_\tau)\, d\tau} / Y_o = y]$$

This is the conditional Laplace transform of

$$t\, Z_v = \int_o^t \varphi(Y_\tau)\, d\tau$$

and the nonconditional Laplace transform is

$$\Phi_t(\lambda) = E[e^{-\lambda \int_0^t \varphi(Y_\tau)\, d\tau}] = E[H_t(\lambda, y)]$$

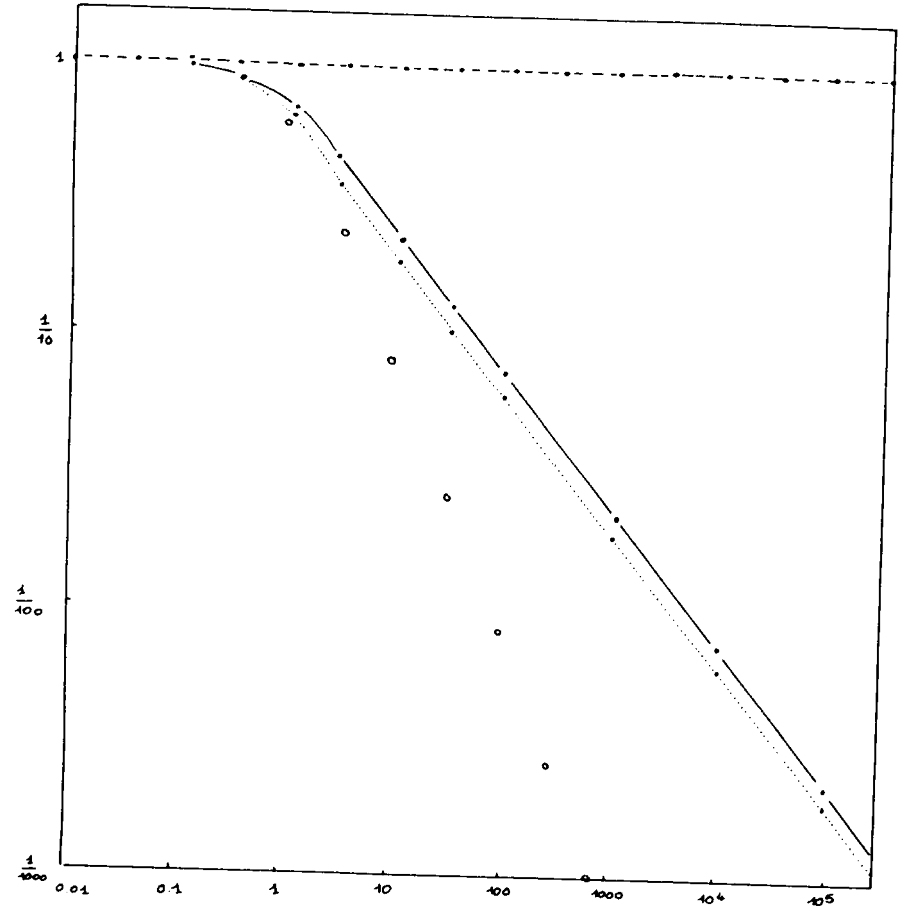

Fig. 4. –The lognormal process in the long run. Dashed line/dot line represents true values of $\chi_3(t)/\sigma^3(t)$: $\sim 1/(t)^{1/2}$ if $t \geqslant 10$. Large dot/small dot line equals isofactorial predictions $\sim 1/t^{1/2}$ if $t > 10$ and isofactorial/true value $\simeq 0.78$. Short dash/dotted line equals affine correction (remains constant !). Dotted line equals the relative variance $\sigma^2(t)/\sigma^2(o) \sim 1/t$ if $t \to \infty$.

In fact, $H_t(\lambda; y)$ is the solution to the equation

$$\partial H_t/\partial t = AH_t - \lambda\varphi(y)\, H_t \qquad (H_o = 1)$$

We consider the diffusion process defined by

$$A = y(d^2/dy^2) + (\alpha - y)(d/dy)$$

This is the *gamma process*; the pdf is the gamma (α) pdf.

As a first example, we choose $\varphi(y) = y$, and the variable of interest is

$$Y_v = \frac{1}{t}\int_o^t Y(\tau)\, d\tau$$

where $Y(t)$ is the gamma process.

By solving the equation

$$(\partial H_t/\partial t) = y(\partial^2 H_t/\partial^2 y) + (\alpha - y)(\partial H_t/\partial y) - \lambda y H_t \qquad (H_o = 1)$$

we find

$$\Phi_t(\lambda) = E[e^{-\lambda Y_v t}]$$
$$= \{Re^{t/2}/[R \text{ ch } (Rt/4) + \text{sh } (Rt/4)]\,[\text{ch } (Rt/4) + R \text{ sh } (Rt/4)]\}^\alpha$$
$$R = (1 + 4\lambda)^{1/2} \tag{29}$$

(sh and ch denote the hyperbolic sine and cosine). This is the *rigorous solution*.

In this particular case, $y - m$ is a *factor* for the gamma process, so that *the isofactorial and A.C. models coincide* and give the same prediction

$$\Phi_t^*(\lambda) = e^{-\lambda\alpha t(1-\rho)}\; 1/[1 + \lambda\rho t]^\alpha$$

Numerically, Φ_t and Φ_t^* are very similar.

Figure 5 shows the relative third cumulant $\chi_3(t)/\sigma^3(t)$ for this. At the beginning, the true values decrease very slowly because $y - m$ is a factor. The common isofactorial and A.C. prediction is χ_3/σ^3 = constant. It is good if $t \leqslant 2$ or 2.5, or $\sigma^2(t) > \frac{1}{2}\sigma^2(o)$.

In the same figure we give the values of $\chi_3(t)/\sigma^3(t)$ in the case of the Ambarzumian process. It is as follows

(exponential decrease and, from time to time, according to a Poisson process, a random exponentially distributed jump). The Ambarzumian process

- is Markov
- has the same stationary gamma (α) distribution
- has the same exponential covariance $e^{-|h|}$

But $y - m$ is *not* a factor for the Ambarzumian process. This is the reason why the relative cumulant decreases much faster.

Figure 6 shows the same process in the long run; the common prediction of

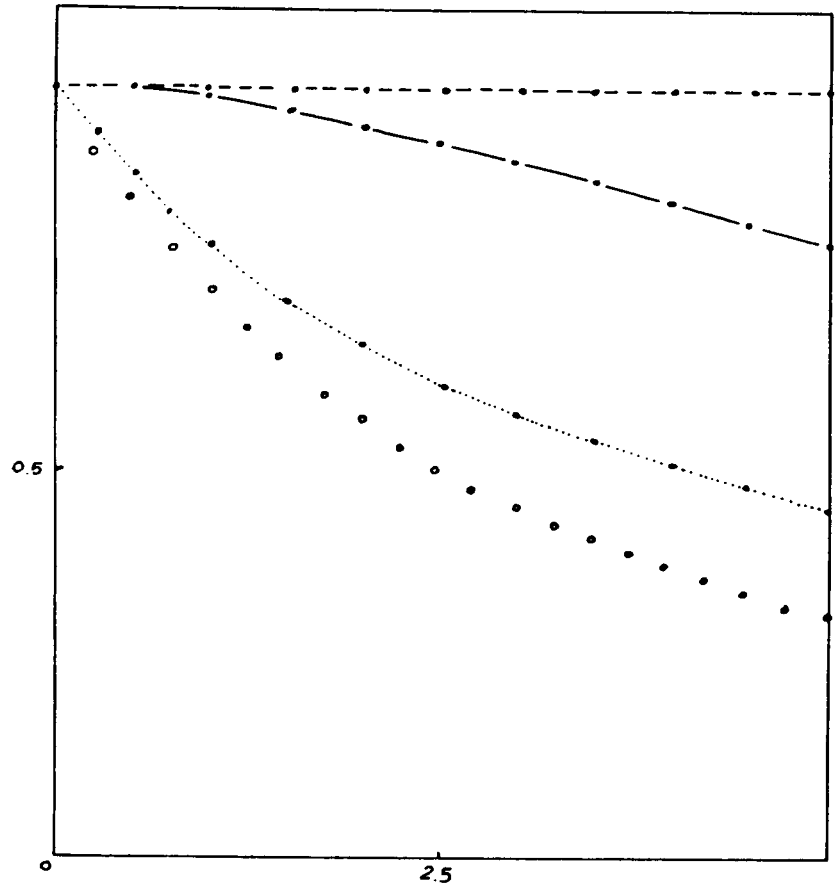

Fig. 5. The gamma process Y_t–order 3 relative cumulant $\chi_3(t)/\sigma^3(t)$ of the variable $Y_V = (1/V)\int_0^t Y(\tau)\,d\tau$. Long dash/dot line equals the true values. Short dash/dot line equals the common prediction of the isofactorial and A.C. models; large dotted line represents the relative variance. Small/large dotted line equals χ_3/σ^3 in the case of the Ambarzumian process (it is *Markov*, has *the same gamma distribution* and *the same covariance* $e^{-|h|}$: but $y - m$ is *not* a factor for the Ambarzumian process).

A.C. and isofactorial models becomes very poor after $t = 5$. The reason is the *convergence of the true variable* Y_v *toward normality*.

The *true* Laplace transform of Y_v, given by (29), has a Weierstrass-type expansion as an infinite product

$$\Phi_t(\lambda) = \prod_{n=1}^{\infty} \left(\frac{b_n}{b_n + \lambda} \right)^{\alpha}$$

i.e., Y_v is an infinite sum of independent R.V. (Matheron, 1983b). The values of

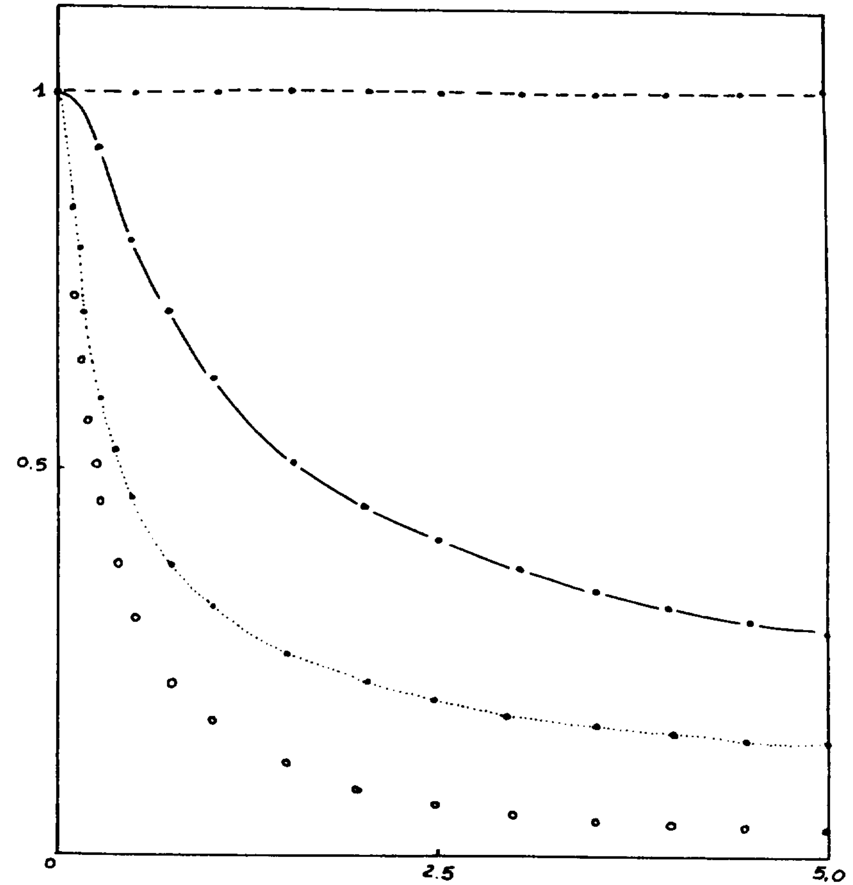

Fig. 6. The gamma process Y_t in the long run–order 3 relative cumulant $\chi_3(t)/\sigma^3(t)$ of the variable $Y_V = 1/t\int_0^t Y(\tau)\,d\tau$. Long dash/dot line equals true values. Short dash/dot line represents the common prediction of isofactorial and A.C. models. Large dot line equals the relative variance $\sigma^2(t)/\sigma^2(o)$. Large/small dotted line is χ_3/σ^3 in the case of the Ambarzumian process.

the parameters b_n are known. In the case

$$\alpha = 1 \qquad \text{i.e.}$$

$$g(y) = e^{-y}$$

we obtain an expansion of the form

$$\Phi_t(\lambda) = \sum \frac{C_n}{b_n + \lambda}$$

so that an explicit inversion is possible. Numerical values are given in Matheron

(1983b): the agreement between the true distribution and the isofactorial prediction is extraordinarily good.

4.6. The Square Y_t^2 of the Gamma Process

With the same gamma diffusion process Y_t, we now choose

$$Z_t = Y_t^2/m_2$$

$$Z_v = \frac{1}{t}\int_o^t \frac{Y_\tau^2}{m_2}\,d\tau$$

Here, $\varphi(y) = y^2/m_2$ is *not a factor*, and so we have *a clear discrimination between the isofactorial and A.C. models.*

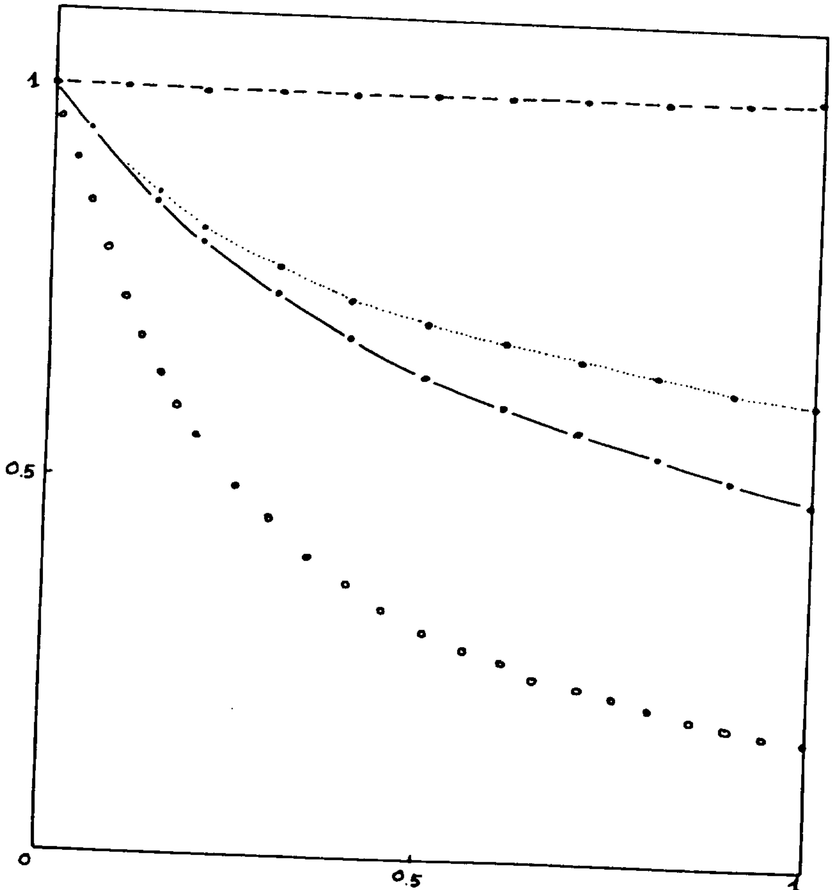

Fig. 7. The gamma diffusion process Y_t, $\alpha = 5$. Relative 3rd cumulant of $Z_V = 1/t \int_o^t Y_\tau^2/m_2\, d\tau$. Large dash/dotted line equals true values. Large and small dotted line represents the isofactorial predictions. Short dash/dotted line equals A.C. predictions (*remain constant*). Dotted line represents the relative variance $\sigma^2(t)/\sigma^2(o)$.

Figure 7: by using Laguerre polynomials, it is possible (although fairly tedious!) to calculate the true values of the third moment. This shows that

- A.C. is *very poor*
- The isofactorial model is very good for $\sigma_t^2 \geq \sigma_o^2/2$, but it is not correct if t is large; in this model, the variable $\varphi_v(Y)$ converges toward the first factor, which is gamma and not normal as it should.

5. A MORE GENERAL MODEL

In the discontinuous cases, the preceding models do not work. Let us give two simple examples.

First Example: Multigaussian Case

If the transform φ is not continuous, our approximation formulas are no longer valid. For instance, if φ is an indicator function, i.e.

$$Z(x) = 1_{Y(x)>a} = 1 \quad \text{if} \quad Y(x) > 1, \qquad 0 \quad \text{otherwise}$$

we find

$$M_2(v) = 1 - G(a) - e^{-a^2/2}/\pi^{1/2}\,\overline{\gamma^{1/2}} + o(\overline{\gamma^{1/2}})$$
$$M_3(v) = 1 - G(a) - \tfrac{3}{2}e^{-a^2/2}\,\overline{\gamma^{1/2}} + o(\overline{\gamma^{1/2}})$$

where G is the standard normal cdf. Note the occurrence of the *square root of the variogram*, which is characteristic of the indicator function.

If $Y(t)$ is the standard Gaussian Markovian process, it is possible to calculate the general moment $M_n(t)$ of the variable

$$Z_v = \frac{1}{t}\int_o^t 1_{Y(\tau)>a}\,d\tau$$

After some fairly difficult calculations, we find

$$M_n(t) = 1 - G(o) - \frac{t^{1/2}}{\pi}e^{-a^2/2}\left[\frac{4n}{1+2n} - 2^{n+1}\frac{(n!)^2}{(2n+1)!}\right] + o(t^{1/2})$$

This is compatible with neither the isofactorial nor the affine correction model.

Second Example: Feller-Type Markov Processes

These processes are *jump processes*, defined by an infinitesimal generator A of the form

$$(A\varphi)(y) = -C(y)\,\varphi(y) + C(y)\int \pi_y(du)\,\varphi(u)$$

where $\pi_y(du)$ is a transition probability, and $C(y) \geq 0$.

A Feller-type process is discontinuous; nevertheless, it is possible to find the first-order approximation for the variable

$$Z_v = \frac{1}{t} \int_o^t \varphi(Y_\tau)\, d\tau$$

For, if t is very small

- either Y_τ *remains constant* on $(0, t)$, with a probability $1 - tC(y) + o(t)$
- or it has *only one jump*, with a probability $tC(y) + o(t)$
- the probability of having *more* than one jump is $o(t)$

Now this (eventual) unique jump point is *uniformly distributed* on $(0, t)$, so that, neglecting events of probability $o(t)$ we have

$$Z_v = UZ_o + (1 - U)\, Z_t$$

with U *uniformly* distributed on (0, 1) and independent of the process. Clearly, this relation also holds if there is no jump, because in this case $Z_o = Z_t$.

This very simple result leads to

$$M_n(v) = M_n(0) + \frac{t}{n+1} \sum_{k=1}^{n-1} E[\varphi^k A \varphi^{n-k}] + o(t)$$

It turns out that this formula also holds in the case of the Ito-type processes and for various other types of processes.

This suggests a new model, more general than the isofactorial model.

A New Model: The Uniform Mixture

If the bivariate distributions $(Z(x), Z(y))$ are of the same type $g_\rho(z, z')$ except for the value of (say) the correlation coefficient ρ, put

$$Z_v = UZ_o + (1 - U)\, Z_1$$

U is uniformly distributed on (0, 1) and independent of Z_o, Z_1. The distribution of (Z_o, Z_1) if $g_\rho(z_o, z_1)$ and ρ is chosen so that the variance has the correct value

$$\sigma_v^2 = ((2 + \rho)/3)\, \sigma^2$$

In practice, this model is good only if $\rho \geqslant 0$. Hence the limitation

$$\sigma_v^2 \geqslant \tfrac{2}{3} \sigma^2$$

If so, this uniform mixture model is practically equivalent to the isofactorial model, in the case of a diffusion-type R.F. (and identical to it for the first-order approximation). But it may be applied in other cases, in particular in *discontinuous* cases. New estimation techniques could be based upon it.

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