RANDOM CLOSED SETS

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E is a locally compact and separable (LCS) space, and $\mathcal{F}$, $\mathcal{G}$ $\mathcal{K}$ respectively the families of the closed, open and compact subsets in E. For any $B \subset E$, $\mathcal{F}_B = \{F,\ F \in \mathcal{F},\ F \cap B \neq \emptyset\}$ is the family of the closed sets hitting B, and $\mathcal{F}^B$ the complementary family. Then the topology on $\mathcal{F}$ generated by the $\mathcal{F}_G$, $G \in \mathcal{G}$ and $\mathcal{F}^K$, $K \in \mathcal{K}$ is compact and separable. The corresponding borelian tribe σ_f is generated either by the $\mathcal{F}_K$, $K \in \mathcal{K}$ or by the $\mathcal{F}_G$, $G \in \mathcal{G}$. A random closed set (RACS) is defined by a probability P on σ_f. The subspace $\mathcal{F}' \subset \mathcal{F}$ of the non empty closed sets is LCS, but non compact (except if E is compact) and we may define σ-finite measures on $\mathcal{F}'$.

If ψ is a function on $\mathcal{K}$, there exists a probability P on (resp. a positive σ-finite measure θ on $\mathcal{F}'$) such that $P(\mathcal{F}_K) = \psi(K)$ $(\theta(\mathcal{F}_K) = \psi(K))$, $K \in \mathcal{K}$ if and only if ψ is an alternating Choquet capacity such that $0 \leq \psi \leq 1$ (resp. $0 \leq \psi < \infty$) (Choquet Theorem). Various classes of RACS are considered and characterized by simple properties of the functionals ψ.

- A RACS A without fixed points is <u>infinitely divisible</u> for the union (is an IDRACS) if and only if it is the union of a Poisson process σ-finite on $\mathcal{F}'$, or if and only if $P(\mathcal{F}^K) = \exp(-\psi(K))$ for a finite Choquet capacity ψ. In the euclidean space $E = R^N$, a RACS A is stable if for any $n > 0$, the union $A_1 \cup \ldots \cup A_n$ of n independent RACS equivalent to A is equivalent to an homothetics of A. A stable RACS is an IDRACS. Then A is <u>stable</u> for the union if and only if the capacity ψ is homogeneous.

- ψ is <u>C-additive</u> if $\psi(K) + \psi(K') = \psi(K \cup K') + \psi(K \cap K')$ for any $K, K' \in C(\mathcal{K})$ (i.e. compact and convex) such that $K \cup K' \in C(\mathcal{K})$. Then, a capacity ψ is C-additive if and only if the corresponding σ-finite measure θ is concentrated on the space $C(\mathcal{F}')$ of the non-empty convex closed sets.

- A RACS is semi-markovian if $A \cap K$ and $A \cap K'$ are independent conditionnally if $A \in \mathcal{F}^C$ for any K, K', $C \in \mathcal{K}$ such that K and K' are separated by C. Then an IDRACS A is semi-markovian (is a SMIDRACS) if and only if it is the union of a Poisson process concentrated on $C(\mathcal{F}')$, i.e. if and only if ψ is C-additive.

- Simple examples of SMIDRACS are : the boolean schemes (union of a Poisson process on $C(\mathcal{K}')$), and the Poisson flats (union of Poisson processes on the linear manifolds).

- More generally, any stationary SMIDRACS is the union of a σ-finite Poisson process concentrated on the convex cylinders with compact bases. The general form of the associated capacities ψ (i.e. of the C-additive capacities invariant under translations) is :

$$\psi(K) = E[\lambda_N(A_o \oplus K)] + \sum_{k=1}^{N} \int_{\mathcal{S}_k} F_k(dS)\; E[\lambda_{N-k}(A_S \oplus \Pi_{S^\perp} K)]$$

where λ_{N-k} is the Lebesgue measure on $\mathbb{R}^{N-k}$, A_o is a RACS a.s. compact and convex, $\mathcal{S}_k \subset C(\mathcal{F}')$ is the set of the k-dimensional subspaces of $\mathbb{R}^N$, F_k a positive measure on $\mathcal{S}_k$, $\Pi_{S^\perp} K$ is the projection of K on the subspace $S^\perp$ orthogonal to $S \in \mathcal{S}_k$, and, for F_k almost every $S \in \mathcal{S}_k$, A_S is a RACS a.s. convex and compact in $S^\perp$. In particular, a RACS is a Poisson flats if and only if it is stable, stationary and semi-markovian.