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DILATIONS ON TOPOLOGICAL SPACES

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2.0 - ALGEBRAIC FRAME.

If E is an arbitrary set, let Γ be a mapping from E into $\mathcal{P}(E)$, i.e. the transform of a point $x \in E$ is a subset $\Gamma(x)$ of E. Then, the mapping Γ admits an extension by a mapping from $\mathcal{P}(E)$ into itself defined as follows :

$$(2.1) \qquad \Gamma(A) = \bigcup_{x \in A} \Gamma(x) \qquad (A \in \mathcal{P}(E))$$

The mapping (1) is called a dilation, see [1], and the notation $\mathcal{S}(E)$, or simply $\mathcal{S}$ will denote the class of the dilations from $\mathcal{P}(E)$ into itself. Obviously, a mapping Γ from $\mathcal{P}(E)$ into itself is a dilation if and only if we have

$$(2.2) \qquad \Gamma(\cup A_i) = \cup \Gamma(A_i)$$

for any family A_i of subsets of E.

As in [1], for any mapping $\Gamma : E \to \mathcal{P}(E)$, we shall define its reciprocal mapping $\check{\Gamma}$ by writing :

$$(2.3) \qquad y \in \check{\Gamma}(x) \text{ if and only if } x \in \Gamma(y) \qquad (x, y \in E)$$

and the corresponding dilation $\check{\Gamma}$, defined according to rule (1), is said to be the reciprocal dilation of Γ. The dilation Γ and its reciprocal $\check{\Gamma}$ satisfy the equivalence :

$$(2.4) \qquad \Gamma(A) \cap B \neq \emptyset \Leftrightarrow A \cap \check{\Gamma}(B) \neq \emptyset$$

for any subsets A, B of E. In fact, the class $\mathcal{S}$ of the dilation is fully characterized by relation (2.4) :

<u>THEOREM 2.1</u> - A mapping Γ from $\mathcal{P}(E)$ into itself is a dilation if and only if there exists a mapping $\check{\Gamma} : \mathcal{P}(E) \to \mathcal{P}(E)$ such that the equivalence (2.4) is true. If so, this reciprocal mapping $\check{\Gamma}$ is unique. Moreover $\check{\Gamma}$ is a dilation and its reciprocal mapping $\check{\check{\Gamma}}$ is Γ itself.

The condition is necessary, because the equivalence (4) follows from definition (3). Conversely, let Γ and $\check{\Gamma}$ be two mappings satisfying equivalence (4). If the set A is a point $x \in E$, i.e. $A = \{x\}$, we write $\Gamma(x)$ instead of $\Gamma(\{x\})$. We must prove that the mapping $x \to \Gamma(x)$ from E into itself satisfies relation (2.1).

If $B = \{x\}$ for a point $x \in E$, it follows from (4) :

$$x \in \Gamma(A) \Leftrightarrow A \cap \check{\Gamma}(x) \neq \emptyset \tag{2.5}$$

Moreover, if $A = \{y\}$ for another point $y \in E$, we get $x \in \Gamma(y)$ if and only if $y \in \check{\Gamma}(x)$, i.e. relation (3). By this relation (2.3) the relation $A \cap \check{\Gamma}(x) \neq \emptyset$, i.e. "there exists a point $y \in A$ such that $y \in \check{\Gamma}(x)$" is equivalent to "there exists a point $y \in A$ such that $x \in \Gamma(y)$", i.e. to $x \in \cup \{\Gamma(y),\ y \in A\}$. Thus equivalence (2.5) implies :

$$\Gamma(A) = \bigcup_{y \in A} \Gamma(y)$$

and Γ is a dilation. In the same way, we find

$$\check{\Gamma}(B) = \bigcup_{y \in B} \check{\Gamma}(y)$$

so that $\check{\Gamma}$ is a dilation, and its uniqueness follows from relation (2.3). The remainder is obvious.

2.1 - UPPER SEMI-CONTINUITY.

From now on, we assume E to be a topological space locally compact, Hausdorff and separable (LCS), and we define certain sub-classes of dilations with good topological properties. If A is a subset of $\mathcal{P}(E)$, $\mathcal{F}_A$ denotes the class of the closed subsets of E which meet A. In the same manner, $\mathcal{K}_A$ denotes the class of compact sets which meet A. The spaces $\mathcal{F} = \mathcal{F}(E)$ and $\mathcal{K} = \mathcal{K}(E)$ are topologized in the usual way (see [2]). Then :

LEMMA 2.1 - For any $A \in \mathcal{P}(E)$, $\mathcal{F}_A$ is closed in $\mathcal{F}$ if and only if A is compact, $\mathcal{K}_A$ is closed in $\mathcal{K}$ if and only if A is closed.

Proof - If A is compact, $\mathcal{F}_A$ is closed in $\mathcal{F}$, by definition of the topology on $\mathcal{F}$. Conversely, let us suppose that $\mathcal{F}_A$ is closed in $\mathcal{F}$. For any $x \notin A$, we have $F = \{x\} \notin \mathcal{F}_A$, and thus there exists a neighborhood of $\{x\}$ in $\mathcal{F}$ of the form

$$\mathcal{F}^K_{G_i \ldots G_N} = \{F : F \in \mathcal{F},\ F \cap K = \emptyset,\ F \cap G_i \neq \emptyset\}$$

containing $\{x\}$ and disjoint from $\mathcal{F}_A$. But in this case $\mathcal{F}^K$ is also disjoint from $\mathcal{F}_A$ and contains $\{x\}$, because $F \cap A = \emptyset$ implies $F' \cap A = \emptyset$ for any other closed set $F' \subset F$, and we have $y \notin A$ for any point $y \notin K$, i.e. $A \subset K$. Thus, for any $x \notin A$, there exists a compact set K_x such that $A \subset K_x$, $x \notin K_x$. It follows that $A = \cap \{K_x,\ x \notin A\}$, and A is compact. The proof is the same for $\mathcal{K}_A$.

LEMMA 2.2 - For any dilation $\Gamma \in \mathcal{S}(E)$, the two following conditions are equivalent :

(2.6) $\Gamma(K)$ is <u>compact</u> for any compact set K

(2.6') $\{F : F \in \mathcal{F},\ \check{\Gamma}(F) \cap K \neq \emptyset\}$ is closed in $\mathcal{F}$ for any $K \in \mathcal{K}$.

In the same way, the two following conditions are equivalent :

(2.7) $\check{\Gamma}(F)$ is closed for any closed set F

(2.7') $\{K : K \in \mathcal{K}, \Gamma(K) \cap F \neq \emptyset\}$ is closed in $\mathcal{K}$ for any $F \in \mathfrak{F}$.

Proof - By Lemma 1, $\Gamma(K)$ is compact if and only if the set

$$\mathfrak{F}_{\Gamma(K)} = \{F : F \in \mathfrak{F},\ F \cap \Gamma(K) \neq \emptyset\}$$

is closed in $\mathfrak{F}$. But, by relation (2.4) , this is equivalent to (2.6') . In the same way, $\check{\Gamma}(F)$ is closed if and only if $\mathcal{K}_{\check{\Gamma}(F)}$ is closed in $\mathcal{K}$, and this is equivalent to (2.7').

THEOREM 2.2 - For any dilation $\Gamma \in \mathcal{S}(E)$, the three following conditions are equivalent :

a) $\Gamma(K)$ is compact for any $K \in \mathcal{K}$ and $\check{\Gamma}(F)$ is closed for any $F \in \mathfrak{F}$.

b) Γ is an u.s.c. mapping from $\mathcal{K}$ into itself

c) $\check{\Gamma}$ is an u.s.c. mapping from $\mathfrak{F}$ into itself.

Proof - This is an immediate consequence of Lemma 2.2 because we have :

a) $\Leftrightarrow$ (2.6) and (2.7)

b) $\Leftrightarrow$ (2.6) and (2.7')

c) $\Leftrightarrow$ (2.6') and (2.7)

Stronger results can be obtained by using the following Lemma :

LEMMA 2.3 - For any dilation $\Gamma \in \mathcal{S}(E)$, $\check{\Gamma}(x)$ is closed for any point $x \in E$ if and only if, for any $A \in \mathcal{P}(E)$, we have :

i) $\Gamma(A) = \cap \{\Gamma(G),\ G \in \mathcal{G},\ G \supset A\}$

In the same way, if $\Gamma' = \complement \check{\Gamma} \complement$ is the erosion dual of $\check{\Gamma}$, the set $\Gamma(x)$ is compact for any point $x \in E$ if and only if we have for any $A \in \mathcal{P}(E)$:

ii) $\Gamma'(A) = \cup \{\Gamma'(K),\ K \in \mathcal{K},\ K \subset A\}$

Proof - The condition i) expresses that for any $x \notin \Gamma(A)$, i.e. $A \cap \check{\Gamma}(x) = \emptyset$, there exists an open set G such that $G \supset A$ and $x \notin \Gamma(G)$, i.e. $G \cap \check{\Gamma}(x) = \emptyset$. With $A = \{y\}$, this implies that any point $y \notin \check{\Gamma}(x)$ admits an open neighborhood G disjoint from $\check{\Gamma}(x)$, i.e. $\check{\Gamma}(x)$ is closed. Conversely, if $\check{\Gamma}(x)$ is closed, the open set $G = \complement \check{\Gamma}(x)$ satisfies the required condition.

If the notation $\mathcal{L}$ denotes the class of the open sets which are the complements of compact sets, i.e. $L \in \mathcal{L}$ if $L = K^c$ for a $K \in \mathcal{K}$, the condition ii) may be rewritten in the form :

ii') $\check{\Gamma}(A) = \cap \{\check{\Gamma}(L),\ L \in \mathcal{L},\ L \supset A\}$

and the proof of the second part of the lemma is the same.

THEOREM 2.3 - A dilation $\Gamma \in \mathcal{S}(E)$ is an u.s.c. mapping of $\mathcal{K}$ into itself if and only if one of the following equivalent conditions is satisfied :

d) The mapping $x \to \Gamma(x)$ is u.s.c. from E into $\mathcal{K}$.

e) $\Gamma(x)$ is compact for any $x \in E$ and $\check{\Gamma}(F)$ is closed for any $F \in \mathcal{F}$.

f) $\check{\Gamma}(y)$ is closed for any $y \in E$ and $\Gamma(K)$ is compact for any $K \in \mathcal{K}$.

Proof - A mapping $x \to \Gamma(x)$ from E into $\mathcal{K}$ is u.s.c. if and only if the set $\{x : x \in E,\ \Gamma(x) \cap F \neq \emptyset\}$ is closed in E for any $F \in \mathcal{F}$. But, by relation (2.4) , this set is $\check{\Gamma}(F)$. Thus the conditions d) and e) are equivalent.

A dilation Γ which maps $\mathcal{K}$ into itself is u.s.c. if and only if $\Gamma = \Gamma_k$, where Γ_k is the u.s.c. closing. of Γ (see [3], section 6). But Γ_k is defined by :

$$\Gamma_k(K) = \cap \{\Gamma(G),\ G \in \mathcal{G},\ G \supset K\}$$

Thus, by lemma 2.3 we have $\Gamma = \Gamma_k$ if $\check{\Gamma}(y)$ is closed for any

$y \in E$. In other words, the condition f) of Theorem 2.3 implies the condition b) of Theorem 2.2 Conversely, the condition a) of Theorem 22 implies the condition f) of Theorem 2.3.

In the same way, an erosion Γ' which maps $\mathcal{G}$ into itself is l.s.c. if and only if $\Gamma' = \Gamma'_g$, where Γ'_g is defined by

$$\Gamma'_g(G) = \bigcup \{\Gamma'(K),\ K \in \mathcal{K},\ K \subset G\}$$

Thus, by Lemma 2.3 if $\Gamma(x)$ is compact for any $x \in E$, we have $\Gamma' = \Gamma'_g$, i.e. Γ' is l.s.c. But the erosion Γ' is l.s.c. from $\mathcal{G}$ into itself if and only if the dual mapping $\check{\Gamma}$ is u.s.c. from $\mathfrak{F}$ into itself. Hence, the condition e) of Theorem 2.3 implies the condition c) of Theorem 2.2. Conversely, the condition a) implies the condition f), and this completes the proof.

Note that, according to [1], any increasing mapping ϕ of $\mathcal{P}(E)$ into itself is an intersection of dilations, or an union of erosions. In fact, if for any $B \in \mathcal{P}(E)$ we define a mapping Γ_B from E into $\mathcal{P}(E)$ by putting :

$$\begin{cases} \Gamma_B(x) = \phi(B) & \text{if } x \in B \\ \Gamma_B(x) = E & \text{if } x \notin B \end{cases}$$

the corresponding dilation $\Gamma_B \in \mathcal{S}(E)$ is defined by :

$$\begin{aligned} \Gamma_B(A) &= \phi(B) && \text{if } A \subset B \\ \Gamma_B(A) &= E && \text{otherwise} \end{aligned} \qquad (A \in \mathcal{P}(E))$$

and we have :

$$\phi(A) = \bigcap \{\Gamma_B(A)\ ;\ B \in \mathcal{P}(E)\}$$

In the same way, <u>any increasing u.s.c. mapping ϕ from $\mathcal{K}$ into $\mathcal{K}$ is an intersection of dilations u.s.c. from $\tilde{\mathcal{K}}$ into $\tilde{\mathcal{K}}$</u> . (for the definition of the space $\tilde{\mathcal{K}} = \mathcal{K} \cup \{E\}$, see [3] section 9.6).

In fact, ψ is u.s.c. if and only if we have for any $K \in \mathcal{K}$:

$$\psi(K) = \cap \{\psi(K'),\ K' \in \mathcal{K},\ \mathring{K}' \supset K\}$$

Thus, for any $K' \in \mathcal{K}$ consider the mapping $\Gamma_{K'}$ from $\tilde{\mathcal{K}}$ into $\tilde{\mathcal{K}}$ defined by

$$\begin{cases} \Gamma_{K'}(K) = \psi(K') & \text{if } K \subset \mathring{K}' \\ \Gamma_{K'}(K) = E & \text{otherwise} \end{cases}$$

Then $\Gamma_{K'}$ is a dilation u.s.c. from $\tilde{\mathcal{K}}$ into itself, and we have

$$\psi(K) = \cap \{\Gamma_{K'}(K)\ ,\ K' \in \mathcal{K}\}$$

for any compact set K.

2.2 - LOWER SEMI-CONTINUITY.

Concerning the lower semi-continuity, we have very similar results. We begin with a lemma :

LEMMA 2.4 - For any $A \in \mathcal{P}(E)$, $\mathcal{K}_A$ is open in $\mathcal{K}$ if and only if A is open in E. $\mathcal{F}_A$ is open in $\mathcal{F}$ if and only if $A \in \mathcal{G}$.

The proof is about the same as for Lemma 2.1.

LEMMA 2.5 - For any dilation $\Gamma \in \mathcal{S}(E)$ and any $G \in \mathcal{G}$, the following conditions are equivalent :

i) $\check{\Gamma}(G)$ is open in E

ii) The set $\{x : x \in E,\ \Gamma(x) \cap G \neq \emptyset\}$ is open in E

iii) The set $\{F : F \in \mathcal{F},\ \Gamma(F) \cap G \neq \emptyset\}$ is open in $\mathcal{F}$

iv) The set $\{K : K \in \mathcal{K},\ \Gamma(K) \cap G \neq \emptyset\}$ is open in $\mathcal{K}$

The equivalence of conditions i) and ii) is an obvious consequence of relation (2.4) . By Lemma 2.4, $\check{\Gamma}(G)$ is open in E if and only if $\mathcal{F}_{\check{\Gamma}(G)}$ is open in $\mathcal{F}$. But, by relation (2.4) , we have

$$\mathfrak{F}_{\check{\Gamma}(G)} = \{F : F \in \mathfrak{F} , \Gamma(F) \cap G \neq \emptyset\}$$

so that conditions i) and iii) are equivalent. In the same way, i) and iv) are equivalent, because the set of condition iv) is $\mathcal{K}_{\check{\Gamma}(G)}$.

THEOREM 2.4 - A dilation $\Gamma \in \mathcal{S}(E)$ is l.s.c. from $\mathcal{K}$ into itself if and only if

$$\check{\Gamma}(G) \in \mathcal{G} \text{ for any } G \in \mathcal{G} \text{ and } \Gamma(K) \in \mathcal{K} \text{ for any } K \in \mathcal{K}.$$

In the same way, Γ is l.s.c. from $\mathfrak{F}$ into itself if and only if

$$\check{\Gamma}(G) \in \mathcal{G} \text{ for any } G \in \mathcal{G} \text{ and } \Gamma(F) \in \mathfrak{F} \text{ for any } F \in \mathfrak{F}.$$

Proof - If the dilation Γ maps $\mathcal{K}$ into itself, Γ is l.s.c. if and only if the set $\{K : K \in \mathcal{K}, \Gamma(K) \cap G \neq \emptyset\}$ is open in $\mathcal{K}$ for any $G \in \mathcal{G}$. But, by Lemma 5, this is equivalent to $\check{\Gamma}(G) \in \mathcal{G}$. The proof of the second statement is exactly the same.

COROLLARY - The dilation Γ is a continuous mapping from $\mathcal{K}$ into itself if and only if one of the two following equivalent conditions is satisfied :

i) The mapping $x \to \Gamma(x)$ is continuous from E into $\mathcal{K}$.

ii) $\Gamma(x) \in \mathcal{K}$ for any $x \in E$, $\check{\Gamma}(G) \in \mathcal{G}$ for any $G \in \mathcal{G}$ and $\check{\Gamma}(F) \in \mathfrak{F}$ for any $F \in \mathfrak{F}$.

In the same way $\check{\Gamma}$ is a continuous mapping from $\mathfrak{F}$ into $\mathfrak{F}$ if and only if

iii) The mapping $x \to \Gamma(x)$ is u.s.c. from E into $\mathcal{K}$ and $\Gamma(G)$ is open for any $G \in \mathcal{G}$.

Proof - If Γ is continuous from $\mathcal{K}$ into itself, condition i) is satisfied. Conversely, if $x \to \Gamma(x)$ is continuous from E into $\mathcal{K}$, Γ is u.s.c. from $\mathcal{K}$ into $\mathcal{K}$, by Theorem 2.3, because $x \to \Gamma(x)$ is u.s.c. But $x \to \Gamma(x)$ is l.s.c. and thus, by Lemma 2.4, $\check{\Gamma}(G)$ is open for any $G \in \mathcal{G}$. Thus, by Theorem 2.4, $K \to \Gamma(K)$ is l.s.c. from $\mathcal{K}$ into $\mathcal{K}$. The proof of the remainder is about the same.

2.3 - OPEN DILATIONS.

We shall say that a dilation $\Gamma \in \mathcal{S}(E)$ is <u>open</u> if $\Gamma(G)$ is open in E for any $G \in \mathcal{G}$. We shall use the following elementary lemma :

<u>LEMMA 2.6</u> - A subset B of E is open in E if and only if $\overline{A} \cap B = \emptyset$ for any $A \in \mathcal{P}(E)$ such that $A \cap B = \emptyset$.

Obviously, this condition is necessary. Conversely, if it is satisfied, put $A = B^c$. Then $B \cap B^c = \emptyset$ implies $B \cap \overline{B^c} = \emptyset$ i.e. $B \subset \mathring{B}$. Thus $B = \mathring{B}$, and $B \in \mathcal{G}$.

<u>Criterion 2.1</u> - For any dilation $\Gamma \in \mathcal{S}(E)$, the following conditions are equivalent :

i) The reciprocal dilation $\check{\Gamma}$ is open

ii) $\Gamma(\overline{A}) \subset \overline{\Gamma(A)}$ for any $A \in \mathcal{P}(E)$

iii) $\overline{\Gamma(\overline{A})} = \overline{\Gamma(A)}$ for any $A \in \mathcal{P}(E)$

iv) The mapping $x \to \overline{\Gamma(x)}$ is l.s.c. from E into $\mathfrak{F}$.

<u>Proof</u> - We always have $\Gamma(A) \subset \Gamma(\overline{A})$, so that ii) implies $\Gamma(A) \subset \Gamma(\overline{A}) \subset \overline{\Gamma(A)}$ and thus $\overline{\Gamma(\overline{A})} = \overline{\Gamma(A)}$. The converse is obvious, therefore the two conditions ii) and iii) are equivalent.

Now, the condition iii) is true if and only if we have for any $G \in \mathcal{G}$ and any $A \in \mathcal{P}(E)$

$$G \cap \Gamma(\overline{A}) = \emptyset \Leftrightarrow G \cap \Gamma(A) = \emptyset$$

i.e., by relation (2.4)

$$\overline{A} \cap \check{\Gamma}(G) = \emptyset \Leftrightarrow A \cap \check{\Gamma}(G) = \emptyset$$

Thus, by Lemma 2.6, condition iii) is satisfied if and only if $\check{\Gamma}$ is open. The equivalence of i) and iv) is obvious.

<u>THEOREM 2.5</u> - A dilation $\Gamma \in \mathcal{S}(E)$ is l.s.c. from $\mathfrak{F}$ into $\mathfrak{F}$ if and only if for any $A \in \mathcal{P}(E)$ we have :

$$\Gamma(\overline{A}) = \overline{\Gamma(A)}$$

<u>Proof</u> - If Γ is l.s.c. from $\mathfrak{F}$ into $\mathfrak{F}$, $\check{\Gamma}$ is open by Theorem 4, and thus $\overline{\Gamma(\bar{A})} = \overline{\Gamma(A)}$ by Criterion 2.1. But $\Gamma(\bar{A}) = \overline{\Gamma(\bar{A})}$, because Γ maps $\mathfrak{F}$ into itself, and thus $\Gamma(\bar{A}) = \overline{\Gamma(A)}$.

Conversely, if $\Gamma(\bar{A}) = \overline{\Gamma(A)}$ for any $A \subset E$, $\Gamma(F)$ is closed for any $F \in \mathfrak{F}$, because $F = \bar{F}$, and $\check{\Gamma}$ is open by Criterion 2.1. Thus, by Theorem 2.4, Γ is l.s.c. from $\mathfrak{F}$ into itself.

<u>THEOREM 2.6</u> - If a dilation $\Gamma \in \mathcal{G}(E)$ is open, then Γ is l.s.c. from $\mathcal{G}$ into itself.

<u>Proof</u> - In the LCS space E, for any open set G there exists an increasing sequence of compact sets K_n such that $K_n \subset \mathring{K}_{n+1}$ and $G = \bigcup K_n$, so that for any dilation Γ we have

$$\Gamma(G) = \bigcup \Gamma(\mathring{K}_n)$$

If the dilation Γ is open, we have $\Gamma(\mathring{K}_n) \in \mathcal{G}$ for each n. Thus, if a compact set K is contained into the union of the increasing sequence of open sets $\Gamma(\mathring{K}_n)$, we have $K \subset \Gamma(\mathring{K}_N)$ for a given number N, and for any other open set G' the inclusion $G' \supset K_N$ implies $\Gamma(G') \supset \Gamma(K_N) \supset \Gamma(\mathring{K}_N) \supset K$. Thus Γ is l.s.c. from $\mathcal{G}$ into $\mathcal{G}$.

Note that an arbitrary increasing mapping from $\mathcal{K}$ into itself is open and u.s.c. if and only if for any $K \in \mathcal{K}$ we have :

$$(2.6) \qquad \psi(K) = \bigcap \{\mathring{\psi}(K'),\ \mathring{K}' \supset K,\ K' \in \mathcal{K}\}$$

In the case of a dilation, we have a stronger result .

<u>THEOREM 2.7</u> - If a dilation Γ maps $\mathcal{K}$ into $\mathcal{K}$, then Γ is open and u.s.c. from $\mathcal{K}$ into $\mathcal{K}$ if and only if we have for <u>any point</u> $x \in E$:

$$(2.7) \qquad \Gamma(x) = \bigcap \{\mathring{\Gamma}(K'),\ K' \in \mathcal{K},\ x \in \mathring{K}'\}$$

<u>Proof</u> - By relation (2.6) , the condition (2.7) is necessary. Conversely, if (2.7) holds and Γ maps $\mathcal{K}$ into itself,

let G be an open set. For any $x \in G$, there exists a compact set K such that :

$$x \in \mathring{K} \subset K \subset G$$

because the space E is locally compact. By (2.7) , this implies

$$\Gamma(x) \subset \mathring{\Gamma}(K) \subset \Gamma(K) \subset \Gamma(G)$$

and therefore we have :

$$\Gamma(G) = \bigcup \{\mathring{\Gamma}(K),\ K \in \mathcal{K},\ K \subset G\}$$

Hence $\Gamma(G)$ is open, as it is the union of open sets $\mathring{\Gamma}(K)$.

In the same way, for any $K' \in \mathcal{K}$ such that $x \in \mathring{K}'$, we have another $K \in \mathcal{K}$ such that $x \in \mathring{K} \subset K \subset \mathring{K}'$, so that (2.7) implies :

$$\Gamma(x) = \bigcap \{\Gamma(K),\ K \in \mathcal{K},\ x \in \mathring{K}\}$$

and there exists a decreasing sequence of compact sets K_n such that $\mathring{K}_n \supset K_{n+1}$, $\bigcap K_n = \{x\}$ and

$$\Gamma(x) = \bigcap \Gamma(K_n) = \bigcap \mathring{\Gamma}(K_n)$$

Now, if we have $\Gamma(x) \subset G$ for an open set, i.e. $\bigcap \Gamma(K_n) \subset G$, this implies that one of these compact sets $\Gamma(K_n)$ is $\subset G$, say $\Gamma(K_N) \subset G$. For any $y \in \mathring{K}_N$, this implies $\Gamma(y) \subset \mathring{\Gamma}(K_N) \subset \Gamma(K_n) \subset G$. Then Γ is u.s.c. from E into $\mathcal{K}$. Hence, by Theorem 3, Γ is u.s.c. from $\mathcal{K}$ into itself.

2.4 - U.S.C. DILATIONS NULL AT INFINITY.

We shall say that a dilation $\Gamma \in \mathcal{P}(E)$ is u.s.c. and null at infinity if Γ is u.s.c. from $\mathcal{K}$ into $\mathcal{K}$ and from $\mathfrak{F}$ into $\mathfrak{F}$.

Note that we have $\Gamma(\emptyset) = \emptyset$ for any dilation Γ, because

$\Gamma(A) = \bigcup \{\Gamma(x),\ x \in A\}$ for any $A \in \mathcal{P}(E)$, and the union of the empty family is the empty set $\emptyset$. Then, if Γ is u.s.c. from $\mathcal{F}$ into $\mathcal{F}$ we have $\Gamma(F_n) \to \emptyset$ for any sequence F_n in $\mathcal{F}$ such that $F_n \to \emptyset$ in $\mathcal{F}$. In particular, if a sequence x_n has no accumulation point in E, the sequence of closed point sets $F_n = \{x_n\}$ converges towards $\emptyset$ in the space $\mathcal{F}$, and we have $\Gamma(x_n) \to \emptyset$ in $\mathcal{F}$. Hence the terminology.

By Theorem 2.2, Γ is u.s.c. and null at infinity if and only if Γ and $\check{\Gamma}$ are u.s.c. from $\mathcal{K}$ into itself (or from $\mathcal{F}$ into $\mathcal{F}$). It follows :

<u>THEOREM 2.8</u> - A dilation $\Gamma \in \mathcal{S}(E)$ is u.s.c. and null at infinity if and only if the reciprocal dilation $\check{\Gamma}$ is u.s.c. and null at infinity.

This class is characterized by

<u>THEOREM 2.9</u> - A dilation $\Gamma \in \mathcal{S}(E)$ is u.s.c. and null at infinity if and only if $x \to \Gamma(x)$ is u.s.c. from E into $\mathcal{K}$ and $\check{\Gamma}(K) \in \mathcal{K}$ for any compact set K.

<u>Proof</u> - These conditions are necessary, because Γ is u.s.c. from $\mathcal{F}$ into $\mathcal{F}$ if and only if $\check{\Gamma}$ is u.s.c. from $\mathcal{K}$ into $\mathcal{K}$ (Theorem 2.2) and this implies $\check{\Gamma}(K) \in \mathcal{K}$ for any compact set K.

Conversely, let us suppose that Γ is u.s.c. from E into $\mathcal{K}$ and $\check{\Gamma}(K) \in \mathcal{K}$ for any $K \in \mathcal{K}$. Then by Theorem 2.3 condition d), Γ is u.s.c. from $\mathcal{K}$ into $\mathcal{K}$. We must prove that Γ is u.s.c. from $\mathcal{F}$ into $\mathcal{F}$ or, which is the same by Theorem 2.2 $\check{\Gamma}$ is u.s.c. from $\mathcal{K}$ into $\mathcal{K}$. Since $\check{\Gamma}(K)$ is compact for any $K \in \mathcal{K}$, by condition f) of Theorem 2.3 we only have to prove that $\Gamma(y)$ <u>is closed</u> for any $y \in E$.

Let $x_n \in \Gamma(y)$ be a sequence such that $x_n \to x$ in E. We must prove that $x \in \Gamma(y)$. For each n, we have $x_n \in \Gamma(y)$, i.e. $y \in \check{\Gamma}(x_n)$. But $\check{\Gamma}$ is u.s.c. from $\mathcal{F}$ into $\mathcal{F}$ because Γ is u.s.c. from $\mathcal{K}$ into $\mathcal{K}$ (Theorem 2.2). Then $y \in \check{\Gamma}(x_n)$ and $x_n \to x$ imply $y \in \check{\Gamma}(x)$, i.e. $x \in \Gamma(y)$. Hence, $\Gamma(y)$ is closed. This completes the proof.

<u>Criterion 2.2</u> - Let $\Gamma \in \mathcal{G}(E)$ be a dilation such that $x \to \Gamma(x)$ is u.s.c. from E into $\mathcal{K}$. Then, Γ is u.s.c. and null at infinity if and only if one of the following equivalent conditions is satisfied :

i) For any compact set K, we have another $K' \in \mathcal{K}$ such that $x \notin K'$ implies $\Gamma(x) \cap K = \emptyset$.

ii) $\check{\Gamma}(K)$ is compact for any compact set K

iii) For any sequence x_n without accumulation point in E, the sequence $\Gamma(x_n)$ converges towards the empty set in the space $\mathcal{F}$.

<u>Proof</u> - By Theorem 2.9 if $x \to \Gamma(x)$ is u.s.c. from E into $\mathcal{K}$, Γ is u.s.c. and null at infinity if and only if condition ii) is satisfied.

It remains to show that the three conditions of criterion 2 are equivalent for any dilation Γ such that $x \to \Gamma(x)$ is u.s.c. from E into $\mathcal{K}$. First, note that Γ is u.s.c. from $\mathcal{K}$ into itself by Theorem 2.3.

By relation (2.4) , condition i) is equivalent to "$x \notin K'$ implies $x \notin \check{\Gamma}(K)$" i.e. to $\check{\Gamma}(K) \subset K'$. Besides, by Theorem 2.2, $\check{\Gamma}(F)$ is closed for any $F \in \mathcal{F}$, because Γ is u.s.c. from $\mathcal{K}$ into $\mathcal{K}$. In particular, $\check{\Gamma}(K)$ is closed. Being closed and contained into the compact set K', $\check{\Gamma}(K)$ is a compact set. Thus, condition i) implies condition ii). Conversely, if ii) holds, we have $\Gamma(x) \cap K = \emptyset$ for any $x \notin K' = \check{\Gamma}(K) \in \mathcal{K}$, and ii) implies i). Hence, the two conditions i) and ii) are equivalent.

Now, let us show that i) implies iii). Let $\{x_n\}$ be a sequence without accumulation point in E. We have $\Gamma(x_n) \to \emptyset$ in $\mathcal{F}$ if, for any $K \in \mathcal{K}$, $\Gamma(x_n) \cap K = \emptyset$ for n large enough. By condition i), there exists $K' \in \mathcal{K}$ such that $\Gamma(x) \cap K = \emptyset$ if $x \notin K'$. But the sequence $\{x_n\}$ has no accumulation point, so that we have $x_n \notin K'$, and thus $\Gamma(x_n) \cap K = \emptyset$ for n large enough. Hence, $\Gamma(x_n) \to \emptyset$ in $\mathcal{F}$, and condition iii) is satisfied.

Suppose now that condition ii) is <u>not</u> true. Then, there exists $K \in \mathcal{K}$ such that $\check{\Gamma}(K)$ is not compact. Nevertheless, $\check{\Gamma}(K)$

is closed, because Γ is u.s.c. from $\mathcal{K}$ into $\mathcal{K}$ (Theorem 2.2). Being closed and not compact, the set $\check{\Gamma}(K)$ contains a sequence $\{x_n\}$ without accumulation point. We have $x_n \in \check{\Gamma}(K)$ for any n, and thus there exists a point $y_n \in K$ such that $x_n \in \check{\Gamma}(y_n)$, i.e. $y_n \in \Gamma(x_n)$. Being contained in the compact set K, the sequence $\{y_n\}$ has an accumulation point y, and this implies $y \in \overline{\lim}\, \Gamma(x_n)$ in $\mathfrak{F}$: thus $\Gamma(x_n)$ does not converge towards $\emptyset$ in $\mathfrak{F}$. We conclude that iii) implies ii). This completes the proof.

<u>THEOREM 2.10</u> - If a dilation Γ is u.s.c. and null at infinity, the three following conditions are equivalent :

i) Γ is continuous from $\mathcal{K}$ into $\mathcal{K}$

ii) Γ is continuous from $\mathfrak{F}$ into $\mathfrak{F}$

iii) $\check{\Gamma}$ is open, i.e. $\check{\Gamma}(G) \in \mathcal{G}$ for any open set G.

<u>Proof</u> - Since Γ is u.s.c. from $\mathcal{K}$ into $\mathcal{K}$ and also from $\mathfrak{F}$ into $\mathfrak{F}$, the equivalence of the three conditions immediately follows from Theorem 2.4.

We say that a dilation Γ is <u>continuous and null at infinity</u> if it is continuous at the same time from $\mathcal{K}$ into $\mathcal{K}$ and from $\mathfrak{F}$ into $\mathfrak{F}$. By Theorem 2.10, Γ is continuous and null at infinity if and only if $\check{\Gamma}$ is open and Γ is u.s.c. and null at infinity.

Do note that a dilation Γ continuous and null at infinity is not open in general, so that the reciprocal dilation $\check{\Gamma}$ is only u.s.c. and null at infinity. If we want to preserve the symmetry between Γ and $\check{\Gamma}$ we must consider the class of the <u>dilations open, continuous and null at infinity</u>.

<u>COROLLARY</u> - A dilation Γ belongs to the class $\mathcal{S}_0$ of the dilations open, continuous and null at infinity if and only if $\check{\Gamma} \in \mathcal{S}_0$.

The following criterion is an obvious consequence of the preceding results and of the corollary of Theorem 2.4 :

Criterion 2.3 - A dilation Γ is open, continuous and null at infinity if and only if the three following conditions are satisfied :

i) $x \to \Gamma(x)$ is continuous from $\mathcal{K}$ into $\mathcal{K}$
ii) $\check{\Gamma}(K)$ is compact for any compact set K
iii) $\Gamma(G)$ is open for any open set G

2.5 - OPENING ASSOCIATED WITH A DILATION.

If Γ is a dilation, we define the associated opening γ as in [1], by writing for any $A \in \mathcal{P}(E)$

$$\gamma(A) = \cup \{\Gamma(x) \; : \; x \in E, \; \Gamma(x) \subset A\}$$

The class $\mathcal{B}_\gamma$ of the sets invariant under γ is the class of the sets B which are $B = \Gamma(A)$ for $A \in \mathcal{P}(E)$, i.e. $\mathcal{B}_\gamma$ is the range $\Gamma(\mathcal{P})$ of the dilation Γ.

If we consider the erosion $\Gamma' = \complement \check{\Gamma} \complement$ dual of the dilation $\check{\Gamma}$, we may write :

$$\gamma = \Gamma \circ \Gamma'$$

It is not generally easy to obtain simple criteria for the various topological properties of the opening γ, and we shall only examine particular cases.

If the reciprocal dilation $\check{\Gamma}$ is open, we know that $\check{\Gamma}$ is l.s.c. from $\mathcal{G}$ into itself (Theorem 2.6), and thus $\Gamma' = \complement \Gamma \complement$ is u.s.c. from $\mathcal{F}$ into $\mathcal{F}$. Moreover, if Γ itself is u.s.c. from $\mathcal{F}$ into $\mathcal{F}$, the opening $\gamma = \Gamma\Gamma'$ is u.s.c. from $\mathcal{F}$ into $\mathcal{F}$ since it is the composition of two increasing u.s.c. mappings. But the opening γ is anti-extensive, so that the closed set $\gamma(K)$ is compact for any $K \in \mathcal{K}$, because we have $\gamma(K) \subset K \in \mathcal{K}$. Moreover, the mapping $K \to \gamma(K)$ from $\mathcal{K}$ into itself is u.s.c. not only for the relative topology on $\mathcal{K}$ deduced from $\mathcal{F}$, but also for the myope topology of $\mathcal{K}$. In fact, if a sequence K_n converges towards K in in $\mathcal{K}$ for the myope topology, the compact sets K_n are contained inside a fixed $K_o \in \mathcal{K}$, and we have :

$$\gamma(K_n) \subset \gamma(K_o) \subset K_o$$

It follows that the sequence $\gamma(K_n)$ has the same accumulation points in $\mathcal{K}$ and in $\mathcal{F}$, and the inclusion $\gamma(K) \subset \overline{\lim}\, \gamma(K_n)$ also holds for the myope topology. Hence :

<u>THEOREM 2.11</u> - If a dilation $\Gamma \in \mathcal{S}(E)$ is <u>continuous</u> from $\mathcal{F}$ into $\mathcal{F}$, the associated opening $\gamma = \Gamma\Gamma'$ is u.s.c. from $\mathcal{F}$ into $\mathcal{F}$ and from $\mathcal{K}$ into $\mathcal{K}$.

This is a simple consequence of the previous considerations, because, by Theorem 2.4, Γ is continuous from $\mathcal{F}$ into $\mathcal{F}$ if and only if $\check{\Gamma}$ is open and Γ is u.s.c. from $\mathcal{F}$ into $\mathcal{F}$.

In the same way :

<u>THEOREM 2.12</u> - If a dilation $\Gamma \in \mathcal{S}(E)$ is <u>open</u> and <u>u.s.c.</u> from $\mathcal{K}$ into $\mathcal{K}$, the associated opening $\gamma = \Gamma\Gamma'$ is l.s.c. from $\mathcal{G}$ into $\mathcal{G}$.

<u>Proof</u> - By Theorem 2, $\check{\Gamma}$ is u.s.c. from $\mathcal{F}$ into $\mathcal{F}$, i.e. Γ' is l.s.c. from $\mathcal{G}$ into $\mathcal{G}$, because Γ is u.s.c. from $\mathcal{K}$ into $\mathcal{K}$. By Theorem 6, Γ is l.s.c. from $\mathcal{G}$ into $\mathcal{G}$, because it is open. Thus $\gamma = \Gamma\Gamma'$ is l.s.c. from $\mathcal{G}$ into $\mathcal{G}$, because it is the composition of two l.s.c. increasing mappings from $\mathcal{G}$ into $\mathcal{G}$.

We recall that an opening γ is said to be <u>compact</u> if the following conditions are satisfied :

i) γ is u.s.c. from $\mathcal{K}$ into itself, u.s.c. from $\mathcal{F}$ into itself and l.s.c. from $\mathcal{G}$ into itself.

ii) γ is the smallest extension to $\mathcal{P}(E)$ of its restriction to $\mathcal{K}$.

Then :

<u>THEOREM 2.13</u> - If a dilation Γ is open, continuous and null at infinity, the associated opening $\gamma = \Gamma\Gamma'$ is compact.

<u>Proof</u> - Here, Γ is continuous from $\mathfrak{F}$ into $\mathfrak{F}$, so that γ is u.s.c. from $\mathfrak{F}$ into $\mathfrak{F}$ and from $\mathcal{K}$ into $\mathcal{K}$ (Theorem 2.11). Moreover, Γ is open and u.s.c. from $\mathcal{K}$ into $\mathcal{K}$ and thus, by Theorem 12, γ is l.s.c. from $\mathcal{G}$ into $\mathcal{G}$. Hence, condition i) is satisfied.

Now, for any $A \subset E$, we have $\gamma(A) = \bigcup \{\Gamma(x),\ x \in E,\ \Gamma(x) \subset A\}$. But $\Gamma(x)$ is compact for any $x \in E$ and $\Gamma(x) = \gamma\ \Gamma(x)$. Hence $\gamma(A) \subset \bigcup \{\gamma(K),\ K \in \mathcal{K},\ K \subset A\}$ and the equality, because the converse inclusion always holds. Thus, condition ii) is satisfied, and the opening γ is compact.

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