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FILTERS AND LATTICES

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4.0 - INTRODUCTION

Throughout this paper, $\mathcal{P}$ will be a complete lattice, that is, a set $\mathcal{P}$ provided with an ordering $<$ such that any finite or infinite family of elements $A_i \in \mathcal{P}$ has a least upper bound $\vee A_i \in \mathcal{P}$ and a greatest lower bound $\wedge A_i \in \mathcal{P}$. In particular, $\mathcal{P}$ itself has a greatest element E and a smallest element $\emptyset$.

In mathematical morphology, the lattice $\mathcal{P}$ will be the set of all the possible images to be treated. For instance, $\mathcal{P}$ may be the set $\mathcal{P}(E)$ of all the subsets of a given space E : in this case, the supremum $\vee$ is the union $\cup$ and the infimum $\wedge$ is the intersection $\cap$. If E is a topological space, $\mathcal{P}$ may be the set $\mathcal{F}(E)$ of the closed subsets of E, or the set $\mathcal{G}(E)$ of the open subsets of E. In the case $\mathcal{P} = \mathcal{F}(E)$, the infimum $\wedge$ is the ordinary intersection $\cap$, because $\mathcal{F}$ is closed under intersection, but the supremum $\vee$ is the closed union $\vee A_i = \overline{\cup A_i}$. In other cases, $\mathcal{P}$ will be a family of functions, for instance the lattice of all the positive functions bounded by a given number b, and so on.

A transform, or mapping $\psi : \mathcal{P} \to \mathcal{P}$ is increasing if $A < A'$ implies $\psi(A) < \psi(A')$. It is idempotent if $\psi \circ \psi = \psi$. We shall say that ψ is a filter if it is increasing and idempotent.

A transform ψ is extensive if $\psi(A) > A$ for any $A \in \mathcal{P}$, that is $\psi > I$, where I is the identity function on $\mathcal{P}$. A transform ψ is anti-extensive if $\psi < I$. A closing is an extensive filter, an opening is an anti-extensive filter.

Let $\mathcal{P}'$ denote the set of all the increasing mappings from $\mathcal{P}$ into itself. The relationship $\psi < \psi'$, i.e. $\psi(A) < \psi'(A)$

for any $A \in \mathcal{P}$ is an ordering on $\mathcal{P}'$, and $\mathcal{P}'$ provided with this ordering is a complete lattice : the supremum $\vee \psi_i$, for instance, is the mapping ψ defined by $\psi(A) = \vee \psi_i(A)$, $A \in \mathcal{P}$. Clearly, $\psi_1 < \psi_1'$ and $\psi_2 < \psi_2'$ imply $\psi_1 \psi_2 < \psi_1' \psi_2'$. More generally :

$$(4.1) \quad \begin{cases} (\vee \psi_i) \circ \psi = \vee (\psi_i \psi) \\ (\wedge \psi_i) \circ \psi = \wedge (\psi_i \psi) \end{cases}$$

Note that the equalities $\psi \circ (\vee \psi_i) = \vee (\psi \psi_i)$ or $\psi \circ (\wedge \psi_i) = \wedge (\psi \psi_i)$ are <u>not</u> true in general. We only have the inequalities

$$(4.2) \quad \begin{cases} \psi \circ (\vee \psi_i) > \vee (\psi \psi_i) \\ \psi \circ (\wedge \psi_i) < \wedge (\psi \psi_i) \end{cases}$$

In particular, $\psi \circ (I \vee \psi)$ is greater than $\psi \vee \psi \psi$, while $(I \vee \psi) \circ \psi = \psi \vee \psi \psi$. If ψ is a <u>filter</u>, we have

$$(I \vee \psi) \circ \psi = \psi \quad ; \quad (I \wedge \psi) \circ \psi = \psi$$

but only :

$$\psi \circ (I \vee \psi) > \psi \quad ; \quad \psi \circ (I \wedge \psi) < \psi .$$

We say that a filter ψ is a <u>$\vee$ - filter</u> (resp. a $\wedge$ - filter) if $\psi \circ (I \vee \psi) = \psi$ (resp. $\psi = I \wedge \psi = \psi$). A filter ψ which is at the same time a $\vee$ - filter and a $\wedge$ - filter will be called a <u>strong filter</u>. <u>Openings and closings are strong filters</u>.

<u>Proof</u> - If φ, for instance, is a closing, i.e. $\varphi > I$, we have $I \vee \varphi = \varphi$ and $I \wedge \varphi = I$, and thus $\varphi \circ (I \wedge \varphi) = \varphi$, $\varphi \circ (I \vee \varphi) = \varphi \varphi = \varphi$.

If γ is an opening and φ is a closing, then $\gamma\varphi$ is a $\vee$ - filter and $\varphi\gamma$ is a $\wedge$ - filter (for the proof, see criterion 4.7 below).

We shall see that the converse is true. In other words, a transform ψ is a $\vee$ - filter (resp. a $\wedge$ - filter) if and only if ψ is of the form $\psi = \gamma\varphi$ (resp. $\psi = \varphi\gamma$) for an opening γ and a closing φ.

Note that filters ψ exist which are neither $\vee$ - nor $\wedge$ - filters, so that they are not of the form $\gamma\varphi$ or $\varphi\gamma$.

EXAMPLE : If $B \in \mathcal{P}$ is neither E nor $\emptyset$, let us consider the transform ψ defined as follows :

$$\psi(A) = A \quad \text{if} \quad A > B \text{ or } A < B$$

$$\psi(A) = B \quad \text{otherwise}$$

Then, ψ is increasing and idempotent, i.e. is a filter. But :

$$\psi\,(A \vee \psi(A)) = A \quad \text{if } A < B \quad , = A \vee B \text{ otherwise}$$

$$\psi\,(A \wedge \psi(A)) = A \quad \text{if } A > B \quad , = A \wedge B \text{ otherwise}$$

Hence, ψ is neither a $\vee$ - filter nor a $\wedge$ - filter.

4.1 - SEVEN CRITERIA

For any mapping $\phi : \mathcal{P} \to \mathcal{P}$, the notation $\mathcal{B}_\phi$ will denote the invariance domain of ϕ, that is the family of the elements $B \in \mathcal{B}$ invariant under ϕ, i.e. such that $\phi(B) = B$. The invariance domain $\mathcal{B}_\phi$ is a subset of the range $\phi(\mathcal{P})$, i.e. $\mathcal{B}_\phi \subset \phi(\mathcal{P})$, and ϕ is idempotent if and only if $\mathcal{B}_\phi = \phi(\mathcal{P})$. The following criteria will be useful in the sequel :

Criterion 4.1 - For any mappings f, g from $\mathcal{P}$ into itself,

$$fg = g \Leftrightarrow g(\mathcal{P}) \subset \mathcal{B}_f$$

Criterion 4.2 - Two mappings ϕ and ϕ' from $\mathcal{P}$ into itself are idempotent and have the same invariance domain $\mathcal{B}_\phi = \mathcal{B}_{\phi'}$, if and only if

$$\phi\phi' = \phi' \quad \text{and} \quad \phi'\phi = \phi$$

Proof - Criterion 4.1 is obvious. For any mapping ϕ, ϕ', we have $\mathcal{B}_\phi \subset \phi(\mathcal{P})$ and $\mathcal{B}_{\phi'} \subset \phi'(\mathcal{P})$, so that the inclusions

(a) $\qquad \phi'(\mathcal{P}) \subset \mathcal{B}_\phi$ and $\phi(\mathcal{P}) \subset \mathcal{B}_{\phi'}$

are satisfied if and only if

$$\mathcal{B}_\phi = \mathcal{B}_{\phi'} = \phi(\mathcal{P}) = \phi'(\mathcal{P})$$

i.e. if ϕ and ϕ' are idempotent and admit the same invariance domain. But the inclusions (a) are equivalent to $\phi\phi' = \phi'$ and $\phi'\phi = \phi$ (criterion 4.1). Criterion 4.2 follows.

Criterion 4.3 - Let ϕ be idempotent on $\mathcal{P}$. Then, for any mapping f from $\mathcal{P}$ into itself such that

$$f\phi = \phi$$

ϕf is idempotent, and $\mathcal{B}_{\phi f} = \mathcal{B}_\phi$.

Proof - If $f\phi = \phi$, we have $\phi f \cdot \phi = \phi\phi = \phi$ and $\phi \cdot \phi f = \phi f$. Thus ϕf is idempotent, and $\mathcal{B}_{\phi f} = \mathcal{B}_{\phi}$ (criterion 4.2).

Criterion 4.4 - Let ϕ be idempotent on $\mathcal{P}$. Then, for any mapping $f = \mathcal{P} \to \mathcal{P}$ such that

$$\phi f = \phi$$

$f\phi$ is idempotent.

Proof - If $\phi f = \phi$, we have $f \phi f \phi = f \phi \phi = f \phi$.

In these four criteria, the ordering $<$ does not intervene. From now on, we will only consider increasing mappings ψ i.e. $\phi \in \mathcal{P}'$. For any filter ϕ, the class of the filters ϕ' which have the same invariance domain $\mathcal{B}_{\phi}$ as ϕ will be denoted $\mathcal{I}d(\mathcal{B})$. In the same way, if $\mathcal{B}$ is a subset of $\mathcal{P}$, $g_d(\mathcal{B})$ will denote the class of the filters ϕ such that $\mathcal{B}_{\phi} = \mathcal{B}$. In general, $\mathcal{J}(\mathcal{B})$ will be empty, and we have to find the condition under which a given subset $\mathcal{B} \subset \mathcal{P}$ is an actual invariance domain.

For any subset $\mathcal{B} \subset \mathcal{P}$, we shall consider the class $\underset{\sim}{\mathcal{B}}$ closed under $\vee$ generated by $\mathcal{B}$, that is the intersection of all the classes closed under $\vee$ and containing $\mathcal{B}$. This class $\underset{\sim}{\mathcal{B}}$ is the invariance domain of an opening which will be denoted $\underset{\sim}{I}_{\mathcal{B}}$, or simply $\underset{\sim}{I}$, if there is no ambiguity. In the same way, $\tilde{\mathcal{B}}$ will denote the class closed under $\wedge$ generated by $\mathcal{B}$ and $\tilde{I}_{\mathcal{B}}$, or simply $\tilde{I}$ will be the corresponding closing. $\underset{\sim}{I}_{\mathcal{B}}$ (resp. $\tilde{I}_{\mathcal{B}}$) is the smallest (resp. the greatest) increasing extension on $\mathcal{P}$ of the identity function $I_{\mathcal{B}}$ on $\mathcal{B}$. Explicitly :

$$\begin{cases} \underset{\sim}{I}(A) = \vee \{B : B \in \mathcal{B} , B < A\} \\ \tilde{I}(A) = \wedge \{B : B \in \mathcal{B} , B > A\} \end{cases}$$

Criterion 4.5 - For any subset $\mathcal{B} \subset \mathcal{P}$ and any $f \in \mathcal{P}'$, the following equivalence is true :

$$\mathcal{B}_f \supset \mathcal{B} \Leftrightarrow \underset{\sim}{I}_{\mathcal{B}} \subset f \subset \tilde{I}_{\mathcal{B}}$$

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Proof - If any $B \in \mathcal{B}$ is invariant under f, f is an increasing extension of $I_{\mathcal{B}}$, and thus is smaller (resp. greater) than the greatest (resp. the smallest) extension of $I_{\mathcal{B}}$. Conversely, we have $\underset{\sim}{I}(B) = \tilde{I}(B) = B$ for any $B \in \mathcal{B}$, so that $\underset{\sim}{I} \subset f \subset \tilde{I}$ implies $f(B) = B$ and $\mathcal{B} \subset \mathcal{B}_f$.

The following criterion is the starting point of the whole theory :

Criterion 4.6 - Let f and g be two filters on $\mathcal{P}$ such that $f > g$. Then :

i) $f > fgf > gf \vee fg > gf \wedge fg > gfg > g$

ii) gf, fg, fgf and gfg are filters, and

$fgf \in \mathcal{J}d(fg) \quad ; \quad gfg \in \mathcal{J}d(gf)$

iii) fgf is the smallest filter greater than $gf \vee fg$, gfg is the greatest filter smaller than $gf \wedge fg$.

iv) The following equivalences are true :

$$\mathcal{B}_{fg} = \mathcal{B}_{gf} \Leftrightarrow \mathcal{B}_{fg} = \mathcal{B}_f \cap \mathcal{B}_g \Leftrightarrow \mathcal{B}_{gf} = \mathcal{B}_f \cap \mathcal{B}_g$$

$$\Leftrightarrow fgf = gf \qquad \Leftrightarrow gfg = fg$$

$$\Leftrightarrow gf > fg$$

Proof - The inequalities i) are obvious. From relationships

$$fg = ff\,fg > fg\,fg > fg\,gg = fg$$

we conclude that fg is a filter. By the dual inequalities, gf also is a filter. Now we have

$$fgf \,.\, fg = fg\,fg = fg$$

$$fg \,.\, fgf = fg\,fg \,.\, f = fgf$$

and thus $fgf \in \mathcal{J}d(fg)$ by Criterion 4.2. In the same way, we find $gfg \in \mathcal{J}d(gf)$, so that ii) is proved.

Now, fgf is a filter (by ii) and $fgf > gf \vee fg$ (by i)). Let ϕ be a filter such that $\phi > fg$ and $\phi > gf$. It follows that $\phi = \phi\phi > fggf = fgf$. Thus, fgf is the smallest filtering upper bound of fg and gf. By duality, gfg is the greatest filtering lower bound of fg and gf, hence iii) is proved.

By criterion 2, we have $\mathcal{B}_{fg} = \mathcal{B}_{gf}$ if and only if

$$fg \cdot gf = fgf = gf \quad \text{and} \quad gf \cdot fg = gfg = fg$$

But in fact, these relations imply each other. For instance, $fgf = gf$ implies $fgf \cdot g = gfg$, i.e. $fg = gfg$. By iii), these relations are equivalent to $gf > fg$.

The inclusions

$$\mathcal{B}_f \cap \mathcal{B}_g \subset \mathcal{B}_{fg} \subset \mathcal{B}_f$$

always hold, so that $\mathcal{B}_{fg} = \mathcal{B}_f \cap \mathcal{B}_g$ if and only if $\mathcal{B}_{fg} \subset \mathcal{B}_g$ i.e. by Criterion 4.1 if and only if $gfg = fg$. This completes the proof.

Criterion 4.7 - Let f and g be two filters on $\mathcal{P}$ and $f > g$. Then :

if f is a $\vee$ - filter, gf anf fgf are $\vee$ - filters

if g is a $\wedge$ - filter, fg and gfg are $\wedge$ - filters

if gf is a $\wedge$ - filter, fgf is a $\wedge$ - filter

if fg is a $\vee$ - filter, gfg is a $\vee$ - filter.

Proof - By criterion 4.6 gf and fgf are filters. Now $f > g$ implies $I < I \vee gf < I \vee fgf < I \vee f$, and thus :

$$f < f \circ (I \vee gf) < f \circ (I \vee fgf) < f \circ (I \vee f)$$

If f is a $\vee$ - filter, we have $f = f \circ (I \vee f)$. The above inequalities become equalities, and thus gf and fgf are $\vee$ - filters. In the same way, if gf is a $\wedge$ - filter, the inequalities

$$fgf > fgf \circ (I \wedge fgf) > f \circ gf \circ (I \wedge gf) = fgf$$

become equalities, and fgf is also a $\wedge$ - filter.

EXAMPLE : If γ is an opening and φ is a closing, i.e. $\gamma < I < \varphi$, γ and φ are strong filters, $\gamma\varphi$ and $\varphi\gamma\varphi$ are $\vee$ - filters, $\varphi\gamma$ and $\gamma\varphi\gamma$ are $\wedge$ - filters. Moreover, if $\gamma\varphi$ is a $\wedge$ - filter and thus a strong filter, $\varphi\gamma\varphi$ is a strong filter. In the same way, if $\varphi\gamma$ is a strong filter, $\gamma\varphi\gamma$ is a strong filter.

4.2 - STRUCTURE OF THE INVARIANCE DOMAIN $\mathcal{B}_\psi$.

If $\mathcal{B}$ is an arbitrary subset of $\mathcal{P}$, in general there exist no filter ψ having $\mathcal{B}$ as its invariance domain, and $\mathfrak{J}d(\mathcal{B}) = \emptyset$. Under which condition is $\mathfrak{J}d(\mathcal{B})$ not empty ? A sufficient condition is that $\mathcal{B}$ be closed under $\wedge$ or under $\vee$, because in this case the closing $\tilde{I}_{\mathcal{B}}$ or the opening $\underset{\sim}{I}_{\mathcal{B}}$ belongs to $\mathfrak{J}d(\mathcal{B})$. We shall see that the necessary and sufficient condition is in a certain sense a generalization of these two particular cases.

Let ψ be a filter, and $\mathcal{B} = \mathcal{B}_\psi$ its invariance domain. By Criterion 5, the opening $\underset{\sim}{I} = \underset{\sim}{I}_{\mathcal{B}}$ and the closing $\tilde{I} = \tilde{I}_{\mathcal{B}}$ satisfy the relationships

$$(4.3) \qquad \underset{\sim}{I} < \psi < \tilde{I}$$

Moreover, by Criterion 4.1 ,the inclusions $\mathcal{B} \subset \underset{\sim}{\mathcal{B}}$ and $\mathcal{B} \subset \tilde{\mathcal{B}}$ imply

$$(4.4) \qquad \underset{\sim}{I}\psi = \tilde{I}\psi = \psi$$

so that, by Criterion 4.3 we also have

$$\psi\, \underset{\sim}{I} \in \mathfrak{J}d(\mathcal{B}) \quad , \quad \psi\, \tilde{I} \in \mathfrak{J}d(\mathcal{B})$$

More precisely $\psi_M = \psi\tilde{I}$ is the greatest element of $\mathfrak{J}d(\mathcal{B})$, and $\psi_m = \psi\underset{\sim}{I}$ is its smallest element.

In fact, by relation (4.4) , we have for instance

$$\phi_M = \phi\tilde{I} = \underset{\sim}{I}\,\phi\tilde{I}$$

and the inequalities (4.3) imply :

$$\underset{\sim}{I}\tilde{I} = \underset{\sim}{I}\,\underset{\sim}{I}\,\tilde{I} \subset \underset{\sim}{I}\,\phi\,\tilde{I} = \phi_M \subset \underset{\sim}{I}\,\tilde{I}\,\tilde{I} = \underset{\sim}{I}\,\tilde{I}$$

Hence $\phi_M = \underset{\sim}{I}\,\tilde{I}$, and in the same way $\phi_m = \tilde{I}\,\underset{\sim}{I}$. But the filter $\underset{\sim}{I}\,\tilde{I}$ only depends on $\mathcal{B}$, and not on the choice of the particular element $\phi \in \mathfrak{J}d(\mathcal{B})$. Hence, we have $\phi_M = \phi\,\tilde{I} \supset \phi$ for <u>any</u> $\phi \in \mathfrak{J}d(\mathcal{B})$, and ϕ_M is the greatest element of $\mathfrak{J}d(\mathcal{B})$. In the same way, $\phi_m = \phi\,\underset{\sim}{I} = \tilde{I}\,\underset{\sim}{I}$ is the smallest element of $\mathfrak{J}d(\mathcal{B})$. Also note that by Criterion 7, $\phi_M = \underset{\sim}{I}\,\tilde{I}$ is a <u>$\vee$ - filter</u>, and ϕ_m is a <u>$\wedge$ - filter</u>.

Now, by applying Criterion 4.6, iv) with $f = \tilde{I}$ and $g = \underset{\sim}{I}$, we find

$$\mathcal{B} = \underset{\sim}{\mathcal{B}} \cap \tilde{\mathcal{B}} \quad ; \quad \underset{\sim}{I}\,\tilde{I} = \tilde{I}\,\underset{\sim}{I}\,\tilde{I} \quad ; \quad \tilde{I}\,\underset{\sim}{I} = \underset{\sim}{I}\,\tilde{I}\,\underset{\sim}{I}$$

By the same Criterion 4.6, these necessary conditions are also sufficient. More precisely, we may summarize our results as follow :

<u>THEOREM 4-1</u> - Let $\mathcal{B}$ be a subset of $\mathcal{P}$. Then $\mathfrak{J}d(\mathcal{B})$ is not empty if and only if the condition

$$\mathcal{B} = \underset{\sim}{\mathcal{B}} \cap \tilde{\mathcal{B}}$$

and one of the three following equivalent conditions are satisfied :

i) $\underset{\sim}{I}\,\tilde{I} > \tilde{I}\,\underset{\sim}{I}$

ii) $\underset{\sim}{I}\,\tilde{I} = \tilde{I}\,\underset{\sim}{I}\,\tilde{I}$

iii) $\tilde{I}\,\underset{\sim}{I} = \underset{\sim}{I}\,\tilde{I}\,\underset{\sim}{I}$

If so, $\mathfrak{J}d(\mathcal{B})$ has a greatest element ϕ_M, which is a $\vee$ - filter, and a smallest element ϕ_m, which is a $\wedge$ - filter. Moreover, we have for any other filter $\phi \in \mathfrak{J}d(\mathcal{B})$:

$$(4.5)\quad \begin{cases} \psi_m = \tilde{I}\,\underset{\sim}{I} = \underset{\sim}{I}\,\tilde{I}\,\underset{\sim}{I} = \psi\,\underset{\sim}{I} \\ \psi_M = \underset{\sim}{I}\,\tilde{I} = \tilde{I}\,\underset{\sim}{I}\,\tilde{I} = \psi\,\tilde{I} \\ \psi = \underset{\sim}{I}\,\psi = \tilde{I}\,\psi \\ \underset{\sim}{I} < \psi_m < \psi < \psi_M < \tilde{I} \end{cases}$$

The same theorem may be restated in a more synthetic language. If $\mathcal{I}d(\mathcal{B}) \neq \emptyset$, let B_i be a family of elements of $\mathcal{B}$. We have $\vee B_i \in \underset{\sim}{\mathcal{B}}$ and thus $\underset{\sim}{I}\,(\vee B_i) = \vee B_i$. By the first relation (4.5), it follows for any $\psi \in \mathcal{I}d(\mathcal{B})$

$$\psi(\vee B_i) = \psi\,\underset{\sim}{I}\,(\vee B_i) = \tilde{I}\,\underset{\sim}{I}\,(\vee B_i)$$

But $\underset{\sim}{I}\,(\vee B_i) = \vee B_i$, and then :

$$\tilde{I}\,(\vee B_i) = \psi\,(\vee B_i) \in \mathcal{B}$$

In the same way, we also find :

$$\underset{\sim}{I}\,(\wedge B_i) = \psi\,(\wedge B_i) \in \mathcal{B}$$

In other words, $\mathcal{B}$ is a <u>complete lattice</u> with respect to the ordering on $\mathcal{B}$ deduced from $<$, i.e. any family B_i in $\mathcal{B}$ has a smallest upper bound $\tilde{I}\,(\vee B_i) \in \mathcal{B}$ and a greatest lower bound $\underset{\sim}{I}\,(\wedge B_i) \in \mathcal{B}$.

Conversely, let us assume that $\mathcal{B}$ is a complete lattice. Thus, for any $A \in \mathcal{P}$, the family $\{B : B \in \mathcal{B},\ B > A\}$ has in $\mathcal{B}$ a greatest lower bound, which is :

$$\underset{\sim}{I}\Big(\wedge\,\{B : B \in \mathcal{B}\,,\ B > A\}\Big) = \underset{\sim}{I}\,\tilde{I}\,(A) \in \mathcal{B}$$

But this implies $\mathcal{B}_{\psi_M} \subset \mathcal{B}$ for the filter $\psi_M = \underset{\sim}{I}\,\tilde{I}$. Conversely, for any $B \in \mathcal{B}$ we have $\underset{\sim}{I}(B) = \tilde{I}(B) = B$, and thus $\psi_M(B) = B$, i.e. $\mathcal{B} \subset \mathcal{B}_{\psi_M}$. We conclude $\mathcal{B}_{\psi_M} = \mathcal{B}$, and $\mathcal{I}d(\mathcal{B})$ is not empty. In other words :

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THEOREME 4-2 - Let $\mathcal{B}$ be a subset of $\mathcal{P}$. Then $\mathfrak{Id}(\mathcal{B})$ is not empty if and only if $\mathcal{B}$ is a complete lattice with respect to the ordering on $\mathcal{B}$ deduced from $<$, i.e.

$$\tilde{I}\,(\vee\, B_i) \in \mathcal{B} \quad ; \quad \underset{\sim}{I}\,(\wedge\, B_i) \in \mathcal{B}$$

for any family B_i in $\mathcal{B}$. If so, we have

$$\tilde{I}\,(\vee\, B_i) = \psi\,(\vee\, B_i) \quad ; \quad \underset{\sim}{I}\,(\wedge\, B_i) = \psi\,(\wedge\, B_i)$$

for any $\psi \in \mathfrak{Id}(\mathcal{B})$.

4.3 - UNDER AND OVER FILTERS

The set $\mathcal{P}'$ of the increasing mappings from $\mathcal{P}$ into itself is itself a complete lattice, so that we may also consider the complete lattice $\mathcal{P}''$ of the increasing mappings from $\mathcal{P}'$ into itself. For instance, the transform $f \to ff$ ($f \in \mathcal{P}'$) is an element of $\mathcal{P}''$, which is called the <u>self-composition</u>. The element of $\mathcal{P}'$ invariant under the self-composition are the <u>filters</u> on $\mathcal{P}$. We shall consider four other elements of $\mathcal{P}''$ and their invariance domains :

<u>Over composition from the left</u> : $f \to (I \vee f) \circ f = f \vee ff$

<u>Under composition from the left</u> : $f \to (I \wedge f) \circ f = f \wedge ff$

<u>Over composition from the right</u> : $f \to f \circ (I \vee f)$

<u>Under composition from the right</u> : $f \to f \circ (I \wedge f)$

The corresponding invariant elements will be called respectively : under filters, over filters, $\vee$-under filters, and $\wedge$-over filters.:

<u>Under filters</u> : $f = (I \vee f) \circ f$, i.e. $ff < f$

<u>Over filters</u> : $f = (I \wedge f) \circ f$, i.e. $ff > f$

<u>$\vee$-under filters</u> : $f = f \circ (I \vee f)$

<u>$\wedge$-over filters</u> : $f = f \circ (I \wedge f)$

$ff(I \vee ff) = ff$

$(\wedge f_i)(I \vee (\wedge f_i)) = (\wedge f_i)$

$>$

$<$ $f_i (I \vee f_i) = f_i$ ∂n

$\psi \psi' = \psi$

f mide fihr $\Rightarrow$ $ff(I \vee ff) < ff(I \vee f) = ff$

cf $ff < ff(I \vee ff)$

f

Any $\vee$-under filter is an under filter, any $\wedge$-over filter is an over filter : if f is for instance a $\vee$-under filter, we have :

$$f = f \circ (I \vee f) > f \vee ff > f$$

and thus $f = f \vee ff$ is an under filter. But the converse is not true : there are under filters which are not $\vee$-under filters.

An element $\phi \in \mathcal{T}'$ is a filter if and only if it is at the same time an under and an over filter. It is a $\vee$-filter if and only if it is at the same time a $\vee$-under filter and an over filter. It is a $\wedge$-filter if and only if it is a $\wedge$-over filter and an under filter. It is a strong filter if and only if it is a $\vee$-under filter and a $\wedge$-over filter.

THEOREM 4-3 - The class of the underfilters (resp. over filters) is closed under $\wedge$ (resp. under $\vee$), self composition, over and under composition from the right and from the left.

Proof - Let for instance $f_j, j \in J$ be a family of underfilters. By the relationships (4.1) and (4.2), we have

$$(\bigwedge_{i \in J} f_i) \circ (\bigwedge_{j \in J} f_j) = \bigwedge_{i \in J} (f_i \circ (\bigwedge_{j \in J} f_j)) < \bigwedge_{i \in J} f_i f_i < \bigwedge_{i \in J} f_i$$

so that $\wedge f_i$ is an under filter.

If f is an underfilter, $ff < f$ implies $ff \circ ff < ff$, so that the self composition ff is an underfilter. By $(I \vee f) \circ f = f$ and $(I \wedge f) \circ f = ff$, the over and under compositions from the left are under filters. In the same way, the relations :

$$f \circ (I \vee f) \circ f \circ (I \vee f) = f \circ ((I \vee f) \circ f) \circ (I \vee f)$$
$$= ff \circ (I \vee f) < f \circ (I \vee f)$$

$$f \circ (I \wedge f) \circ f \circ (I \wedge f) = f \circ ((I \wedge f) \circ f) \circ (I \wedge f)$$
$$= f \circ ff \circ (I \wedge f) < f \circ (I \wedge f)$$

..

prove that the over and under compositions from the right are underfilters.

Concerning the $\vee$-under and $\wedge$-over filters, we only obtain the following results :

THEOREM 4-4 - The class of the $\vee$-under filters (resp. $\wedge$-over filters) is closed under $\wedge$ (resp. under $\vee$), self composition, over and under composition from the left and over composition (resp. under composition) from the right. But in general it is not closed under the under composition (resp. over composition) from the right.

Nevertheless, the class of the $\vee$-under filters is closed under the under composition from the right if the lattice $\mathcal{P}$ is distributive (i.e. $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$ and $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$, as it is generally in the applications. In fact, a weaker hypothesis is sufficient. We say that the lattice $\mathcal{P}$ is modular if the following implication holds for any A, B, C in $\mathcal{P}$:

$$(4.6) \qquad B < A \Rightarrow A \wedge (B \vee C) = B \vee (A \wedge C)$$

Any distributive lattice is modular, but the converse is not true in general.

THEOREM 4-5 - If the lattice $\mathcal{P}$ is modular, $f \circ (I \wedge f)$ is a $\vee$-under filter for any $\vee$-under filter $f \in \mathcal{P}'$, and $g \circ (I \vee g)$ is a $\wedge$-over filter for any $\wedge$-over filter $g \in \mathcal{P}'$.

Let us suppose for instance that f is a $\vee$- under filter and put :

$$h = f \circ (I \wedge f)$$

We must show $h = h \circ (I \vee h)$. Explicitly, we have :

$$h \circ (I \vee h) = f \circ [(I \vee h) \wedge f \circ (I \vee h)]$$

But the inequality $h < f$ implies $f < f \circ (I \vee h) < f \circ (I \vee f) = f$, because f is a $\vee$-under filter, and thus $f \circ (I \vee h) = f$. It follows

$$h \circ (I \vee h) = f \circ [(I \vee h) \wedge f]$$

Now, the lattice $\mathcal{P}$ is modular and $h < f$, so that by relation (4.6) we have

$$(I \vee h) \wedge f = (I \wedge f) \vee h = (I \wedge f) \vee f \circ (I \wedge f) = (I \vee f) \circ (I \wedge f)$$

and thus :

$$h \circ (I \vee h) = f \circ (I \vee f) \circ (I \wedge f)$$

But $f \circ (I \vee f) = f$, because f is a $\vee$-under filter, so that

$$h \circ (I \vee h) = f \circ (I \wedge f) = h$$

Hence, h is a $\vee$-under filter.

<u>THEOREM 4-6</u> - Any under filter (resp. $\vee$-under filter) $f \in \mathcal{P}'$ is the infimum of the filters (resp. strong filters) $\phi > f$. Any over filter (resp. $\wedge$-over filter) $g \in \mathcal{P}'$ is the supremum of the filters (resp. strong filters) $\phi < g$.

Let for instance g be an over filter. For any $A \in \mathcal{P}$, we consider the mapping $\phi_A \in \mathcal{P}'$ defined as follows :

$$\begin{cases} \phi_A(A') = g(A) & \text{if } A' > A \text{ or } A' > g(A) \\ \phi_A(A') = \emptyset & \text{otherwise} \end{cases}$$

Clearly, ϕ_A is a filter. If $A' > A$, we have $\phi_A(A') = g(A) < g(A')$. If $A' > g(A)$, we find $\phi_A(A') = g(A) < g(g(A))$, because g is an over filter. But $g(A) < A'$ implies $g(g(A)) < g(A')$, and then $\phi_A(A') < g(A')$. We conclude $\phi_A < g$ for any $A \in \mathcal{P}$. But $\phi_A(A) = g(A)$ for each $A \in \mathcal{P}$, and thus g is the supremum of the filters ϕ_A , $A \in \mathcal{P}$.

Now, let g be a $\wedge$-over filter, i.e. $g \circ (I \wedge g) = g$. Put :

$$\begin{cases} \phi_A(A') = g(A) & \text{if } A' > A \wedge g(A) \\ \phi_A(A') = \emptyset & \text{otherwise} \end{cases}$$

We have $\phi_A < g$, because g is a $\wedge$-over filter, so that $A' > A \wedge g(A)$ implies $\phi_A(A') = g(A) = g(A \wedge g(A)) < g(A')$. In the same way, it is easy to see that ϕ_A is a strong filter. But $\phi_A(A) = g(A)$ for each $A \in \mathcal{P}$, so that g is the supremum of the family of the strong filters $\phi_A < g$.

4.3.1 - The Lattice of the Filters.

By Theorems 4-3 and 4-6, if $\mathcal{U} \subset \mathcal{P}'$ is the class of the filters, $\underset{\sim}{\mathcal{U}}$ is the class of the over filters and $\tilde{\mathcal{U}}$ is the class of the under filters. Let G and F denote the corresponding opening and closing on $\mathcal{P}'$: for any $\phi \in \mathcal{P}'$, $G\phi$ is the supremum of the over filters $g < \phi$ and $F\phi$ is the infimum of the under filters $f > \phi$. By Criterion 4-7, GF and FG are filters (on the lattice $\mathcal{P}'$). The first condition of Theorem 4-1, i.e. $\mathcal{U} = \underset{\sim}{\mathcal{U}} \cap \tilde{\mathcal{U}}$ is satisfied, and thus GF and FG have the same invariance domain $\mathcal{U}$ (i.e. the set of the filters) if and only if $GF > FG$. We shall see that this inequality is true.

Let ϕ be an element of $\mathcal{P}'$. Put $f = F\phi$. Then f is the supremum of the class $\mathcal{C}$ closed under $\vee$ and self composition generated by ϕ. In fact, the class $\mathcal{C}'$ of the elements $\phi' \in \mathcal{P}'$ smaller than f is closed under $\vee$ and self composition. For $g \subset f$ implies $gg \subset ff \subset f$, because $f = F\phi$ is an under filter, and thus $gg \in \mathcal{C}'$. But ϕ itself belongs to $\mathcal{C}'$. This implies $\mathcal{C} \subset \mathcal{C}'$. Now, let f_0 be the supremum of the class $\mathcal{C}$. We have $f_0 \in \mathcal{C}$, because $\mathcal{C}$ is closed under $\vee$, and thus $f_0 \in \mathcal{C}'$, i.e. $f_0 < f$. On the other hand, $\mathcal{C}$ is closed under self composition, so that $f_0 f_0 \in \mathcal{C}$, and thus $f_0 f_0 \subset f_0$, because f_0 is the supremum of $\mathcal{C}$. Hence, f_0 is an under filter. But this implies $f_0 > F\phi = f$, because $f_0 > \phi$. We conclude $f_0 = f$.

If ϕ is an overfilter, instead of the class $\mathcal{C}'$ we may consider the class $\mathcal{C}''$ of the over filters $\phi'' < f = F\phi$: $\mathcal{C}''$ is closed under $\vee$ by Theorem 4-3, and closed under self composition (as above). We see as above that the supremum of $\mathcal{C}''$ is again $f_o = F\phi = f$. Thus f_o is an under filter, because $f_o = F\phi$, and f_o is an over filter, because $f_o \in \mathcal{C}''$: we conclude that f_o is <u>a filter</u>.

In other words :

<u>THEOREM 4-7</u> - For any increasing mapping $\phi \in \mathcal{P}'$, the closing $F\phi$ is the smallest under filter $> \phi$ and the supremum of the class closed under $\vee$ and self composition generated by ϕ. Moreover, if ϕ is an over filter, then $F\phi$ is a filter.

In the same way, the opening $G\phi$ is the greatest over filter $< \phi$ and the infimum of the class closed under $\wedge$ and self composition generated by ϕ. If ϕ is an under filter, then $G\phi$ is a filter.

Moreover, GF and FG are filters on $\mathcal{P}'$ and have the same invariance domain $\mathcal{U}$, which is the set of the filters on $\mathcal{P}$. More precisely GF is the greatest element of $\mathcal{J}d(\mathcal{U})$ and FG is its smallest element. We have in particular :

$$GF > FG \quad ; \quad FGF = FG \quad ; \quad GFG = FG.$$

From Theorem 4-2 it follows :

<u>COROLLARY</u> - The set $\mathcal{U}$ of the filters on $\mathcal{P}$ is a complete lattice. For any family ϕ_i of filters on $\mathcal{P}$, the smallest filter greater than $\vee\, \phi_i$ is $F(\vee\, \phi_i)$, and the greatest filter smaller than $\wedge\, \phi_i$ is $G(\wedge\, \phi_i)$.

If the filters ϕ_i have the same invariance domain $\mathcal{B}$, we have a more precise result :

<u>THEOREM 4-8</u> - Let $\mathcal{B}$ be a subset of $\mathcal{P}$ such that $\mathcal{J}d(\mathcal{B})$ is not empty. Then, $\mathcal{J}d(\mathcal{B})$ is a complete lattice, and for any family ϕ_i of filters in $\mathcal{J}d(\mathcal{B})$, $F\, o(\vee\, \phi_i)$ and $G\, o(\wedge\, \phi_i)$ belong to

..

$\mathcal{J}d(\sqrt{\mathcal{B}})$. Moreover, we have for any $\phi \in \mathcal{J}d(\sqrt{\mathcal{B}})$

$$F\ o(\vee\ \phi_i) = \phi\ o(\vee\ \phi_i) = \tilde{I}_{\sqrt{\mathcal{B}}}\ o(\vee\ \phi_i)$$

$$G\ o(\wedge\ \phi_i) = \phi\ o(\wedge\ \phi_i) = \underset{\sim}{I}_{\sqrt{\mathcal{B}}}\ o(\wedge\ \phi_i)$$

Proof - For any $\phi \in \mathcal{J}d(\sqrt{\mathcal{B}})$, we have $(\vee\ \phi_i)\ o\ \phi = \vee\ (\phi_i\ \phi) = \phi$ (Criterion 4.2) and thus $\phi\ o(\vee\ \phi_i) \in \mathcal{J}d(\sqrt{\mathcal{B}})$ (Criterion 4.3). For any $A \in \mathcal{P}$, $\vee\ \phi_i(A)$ is an element of $\underset{\sim}{\sqrt{\mathcal{B}}}$, so that we have $\vee\ \phi_i = \underset{\sim}{I}\ o(\vee\ \phi_i)$. By relation (4.5), it follows $\phi\ o(\vee\ \phi_i) = \phi\ \underset{\sim}{I}\ o(\vee\ \phi_i) = \tilde{I}\ \underset{\sim}{I}\ o(\vee\ \phi_i)$. But $\underset{\sim}{I}\ o(\vee\ \phi_i) = \vee\ \phi_i$, and thus

$$\phi\ o(\vee\ \phi_i) = \tilde{I}\ \underset{\sim}{I}\ o(\vee\ \phi_i) = \tilde{I}\ o(\vee\ \phi_i)$$

for any $\phi \in \mathcal{J}d(\sqrt{\mathcal{B}})$. If $\phi = \phi_m = \tilde{I}\ \underset{\sim}{I}$ is the infimum of $\mathcal{J}d(\sqrt{\mathcal{B}})$, we have $\phi_m < \vee\ \phi_i$, and thus $\phi_m\ o(\vee\ \phi_i) < F(\vee\ \phi_i)\ o\ F(\vee\ \phi_i) = F(\vee\ \phi_i)$. On the other hand $\tilde{I}\ o(\vee\ \phi_i) > \vee\ \phi_i$ implies $\tilde{I}\ o(\vee\ \phi_i) > F(\vee\ \phi_i)$, because $F(\vee\ \phi_i)$ is the smallest filter $> \vee\ \phi_i$. Thus $\tilde{I}\ o(\vee\ \phi_i) = F(\vee\ \phi_i)$.

4.3.2 - The Lattice of the Strong Filters

Concerning the $\vee$-under and the $\wedge$-over filters, we obtain very similar results if we assume that the lattice $\mathcal{P}$ is modular. In fact, let $\mathcal{V}_0$ be the class of the strong filters on $\mathcal{P}$, so that $\underset{\sim}{\mathcal{V}}_0$ is the class of the $\wedge$-over filters, $\tilde{\mathcal{V}}_0$ is the class of the $\vee$-under filters, and $\mathcal{V}_0 = \underset{\sim}{\mathcal{V}}_0 \cap \tilde{\mathcal{V}}_0$ (Theorem 4-6). Let $F_\vee$ and $G_\wedge$ denote the corresponding closing and opening : for any $\phi \in \mathcal{P}'$, $F_\vee \phi$ is the smallest $\vee$-under filter $f > \phi$, and $G_\wedge \phi$ is the greatest $\wedge$-over filter $g < \phi$. Now, $G_\wedge\ F_\vee$ and $F_\vee\ G_\wedge$ are filters on $\mathcal{P}'$. But this time we have $G_\wedge\ F_\vee \supset F_\vee\ G_\wedge$ only if the lattice $\mathcal{P}$ is modular, because of Theorem 4-5. Otherwise, the proof is the same as above.

THEOREM 4-9 - For any $\phi \in \mathcal{P}'$, the closing $F_\vee \phi$ is the smallest $\vee$-under filter $> \phi$, and the supremum of the class closed under $\vee$ and over composition from the right generated by ϕ. In the same way, the opening $G_\wedge \phi$ is the greatest $\wedge$-over filter $< \phi$, and the infimum of the class closed under $\wedge$ and under composition from the right generated by ϕ.

Moreover, if the lattice $\mathcal{P}$ is modular, then $F_\vee g$ and $G_\wedge f$ are strong filters for any $\wedge$-over filter g and any $\vee$-under filter f. In this case, the filters $G_\wedge F_\vee$ and $F_\vee G_\wedge$ on $\mathcal{P}'$ have the same invariance domain $\mathcal{O}_0$ which is the set of the strong filters on $\mathcal{P}$, $G_\wedge F_\vee$ is the greatest element of $\mathcal{J}d(\mathcal{O}_0)$ and $F_\vee G_\wedge$ its smallest element. In particular, we have

$$G_\wedge F_\vee > F_\vee G_\wedge \quad ; \quad F_\vee G_\wedge F_\vee = G_\wedge F_\vee \ ; \quad G_\wedge F_\vee G_\wedge = F_\vee G_\wedge$$

Proof - Let us prove only the second part of the Theorem.

Let g be a $\wedge$-over filter. We have $F_\vee g > g$, and thus :

$$F_\vee g > (F_\vee g) \circ (I \wedge F_\vee g) > g \circ (I \wedge g) = g$$

because g is a $\wedge$-over filter. But $F_\vee g \circ (I \wedge F_\vee g)$ is a $\vee$-under filter by Theorem 4-5. Since $F_\vee g$ is the smallest $\vee$-under filter greater than g, it follows :

$$(F_\vee g) \circ (I \wedge F_\vee g) = F_\vee g$$

and $F_\vee g$ is a $\wedge$-over filter. Since $F_\vee g$ is also a $\vee$-under filter, it is a strong filter. The remainder follows from Criterion 4.6.

COROLLARY 1 - If the lattice $\mathcal{P}$ is modular, the set $\mathcal{O}_0$ of the strong filters on $\mathcal{P}$ is a complete lattice. For any family ϕ_i of strong filters, $F_\vee(\vee \phi_i)$ is the smallest strong filter greater than $\vee \phi_i$ and $G_\wedge(\wedge \phi_i)$ is the greatest strong filter smaller than $\wedge \phi_i$.

In this corollary, the assumption of modularity cannot be dropped out. On the contrary, the following corollary holds whatever the lattice $\mathcal{P}$:

COROLLARY 2 - If g is a $\wedge$-over filter (resp. an over filter) Fg (resp. $F_\vee g$) is a $\wedge$-filter (resp. a $\vee$-filter). If f is a $\vee$-under filter (resp. an under filter) Gf (resp. $G_\wedge f$) is a $\vee$-filter (resp. a $\wedge$-filter). The set $\mathcal{U}_\wedge$ (resp. $\mathcal{U}_\vee$) of the $\wedge$-filters (resp. $\vee$-filters) is the common invariance domain of $FG_\wedge$ and $G_\wedge F$ (resp. $GF_\vee$ and $F_\vee G$) and is a complete lattice. Moreover, we have

$$G_\wedge F > F\,G_\wedge \; ; \quad G\,F_\vee > F_\vee G$$

Proof - If g is a $\wedge$-over filter, it is an over filter, and thus Fg is a filter (Theorem 4-7). On the other hand, the class of the under filters is closed under the under composition from the right (Theorem 4-3), so that $(Fg) \circ (I \wedge Fg)$ is an under filter, because Fg is a filter, and hence also an under filter. But $Fg > g$ implies that the under filter $(Fg) \circ (I \wedge Fg)$ is greater than $g \circ (I \wedge g) = g$, and thus greater than Fg itself. It follows $(Fg) \circ (I \wedge Fg) = Fg$, and thus the filter Fg is also a $\wedge$-over filter. Hence Fg is a $\wedge$-filter.

It follows that for any $\psi \in \mathcal{P}'$, $F\,G_\wedge \psi$ is a $\wedge$-filter, because $G_\wedge \psi$ is a $\wedge$-over filter, so that the invariance domain $\mathcal{B}_{FG_\wedge}$ of the filter $F\,G_\wedge \in \mathcal{P}''$ is $\subset \mathcal{U}_\wedge$. But conversely, any $\wedge$-filter is invariant under $F\,G_\wedge$. It follows $\mathcal{B}_{FG_\wedge} \supset \mathcal{U}_\wedge$ and the equality.

Now, we have $F\,G_\wedge < F$, because $G_\wedge$ is an opening, and thus $F\,G_\wedge \psi < F\psi$. But this implies $F\,G_\wedge \psi < G_\wedge\, F\psi$, because the $\wedge$-filter $F\,G_\wedge \psi$ is a $\wedge$-over filter smaller than $F\psi$. It follows $F\,G_\wedge < G_\wedge F$, and, by Criterion 4-6, $F\,G_\wedge$ and $G_\wedge F$ have the same invariance domain $\mathcal{B}_{FG_\wedge} = \mathcal{U}_\wedge$, which is a complete lattice by Theorem 4-2. The remainder of corollary 2 follows by duality.

NOTE : It follows from Theorem 4-6 that $G_\wedge$ is the opening associated with $\mathcal{U}_\wedge$, so that, by relation (4.3)n $F\,G_\wedge$ is the smallest element of $\mathcal{J}d(\mathcal{U}_\wedge)$, and, in the same way, $G\,F_\vee$ is the greatest element of $\mathcal{I}d(\mathcal{O}_\vee)$. But we did not prove that $G_\wedge F$ is the greatest element of $\mathcal{I}d(\mathcal{O}_\wedge)$.

4.4 - CHARACTERIZATION OF THE FOUR ENVELOPES

If an increasing mapping $g \in \mathcal{P}'$ is extensive, it is an over filter because $g > I$ implies $gg > g$, and thus Fg is a filter (Theorem 4-7). Hence, Fg is a closing, since $Fg > g > I$, i.e. the smallest closing $> g$. In the same way, for any increasing mapping $\psi \in \mathcal{P}'$, we see that the smallest closing greater than ψ is $F(I \vee \psi)$. It will be more convenient to use a shorter notation by writing :

$$\hat{\psi} = F(I \vee \psi)$$

What is the invariance domain of this closing $\hat{\psi}$? If φ is a closing such that $\varphi > \psi$, we have $\varphi(A) = A > \psi(A)$ for any $A \in \mathcal{B}_\varphi$. The subset of $\mathcal{P}$ defined by

$$\hat{\mathcal{B}}_\psi = \{A : A \in \mathcal{P} , A > \psi(A)\}$$

will be called the anti-extensivity domain of ψ, and we have $\mathcal{B}_\varphi \subset \hat{\mathcal{B}}_\psi$ for any closing $\varphi > \psi$. Conversely, $\mathcal{B}_\varphi \subset \hat{\mathcal{B}}_\psi$ implies $\psi < I$ on $\mathcal{B}_\varphi$, and thus ψ is smaller than the greatest extension φ of the identity on $\mathcal{B}_\varphi$. In other words,

$$\mathcal{B}_\varphi \subset \hat{\mathcal{B}}_\psi \Leftrightarrow \varphi > \psi$$

But the anti-extensivity domain $\hat{\mathcal{B}}_\psi$ is closed under $\wedge$. For $A_i \in \hat{\mathcal{B}}_\psi$, i.e. $\psi(A_i) < A_i$ implies $\psi(\wedge A_i) < \wedge \psi(A_i) < \wedge A_i$. Thus, the invariance domain of $\hat{\psi}$, which is the greatest class $\mathcal{B}_\varphi$ closed under $\wedge$ such that $\mathcal{B}_\varphi \subset \hat{\mathcal{B}}_\psi$ is $\hat{\mathcal{B}}_\psi$ itself. By duality, we obtain similar results concerning the greatest opening $< \psi$:

<u>THEOREM 4-10</u> - For any $\psi \in \mathcal{P}'$, the anti-extensivity domain $\hat{\mathcal{B}}_\psi = \{\psi < I\}$ is the invariance domain of the smallest closing $\hat{\psi}$ greater than ψ, and the extensivity domain $\check{\mathcal{B}}_\psi = \{\psi > I\}$ is the invariance domain of the greatest opening $\check{\psi}$ smaller than ψ.

<u>Criterion 4-8</u> - Let ψ be an increasing mapping, $\hat{\psi}$ the smallest closing $> \psi$, $\check{\psi}$ the greatest opening $< \psi$. Then

ψ is an under filter	if and only if	$\psi = \hat{\psi}\psi$
ψ is a $\vee$-under filter	"	$\psi = \psi\hat{\psi}$
ψ is an over filter	"	$\psi = \check{\psi}\psi$
ψ is a $\wedge$-over filter	"	$\psi = \psi\check{\psi}$

More generally :

ψ is an under filter	if and only if	$\psi = \varphi g$
ψ is a $\vee$-under filter	"	$\psi = g\varphi$
ψ is an over filter	"	$\psi = \gamma g$
ψ is a $\wedge$-over filter	"	$\psi = g\gamma$

for a given increasing mapping g and a closing $\varphi > g$, or an opening $\gamma < g$.

<u>Proof</u> - ψ is an under filter, i.e. $\psi\psi < \psi$, if and only if for any $A \in \mathcal{P}$, $\psi(A)$ belongs to the anti-extensivity domain $\hat{\mathcal{B}}_\psi$, i.e. if and only if $\hat{\psi}(\psi(A)) = \psi(A)$ (Theorem 4-10), i.e. $\psi = \hat{\psi}\psi$.

On the other hand, the closing $\hat{\psi}$ is a $\vee$-under filter and is greater than ψ, so that we have

$$\psi\hat{\psi} < \psi\hat{\psi} \circ (I \vee \psi\hat{\psi}) < \psi\hat{\psi} \circ (I \vee \hat{\psi}) = \psi\hat{\psi}$$

and thus $\psi\hat{\psi} = \psi\hat{\psi} \circ (I \vee \psi\hat{\psi})$ is a $\vee$-under filter. Hence, $\psi = \psi\hat{\psi}$ implies that ψ is a $\vee$-under filter.

Conversely, if ψ is a $\vee$-under filter, we find :

$$\psi = \psi\check{\psi}$$

$$\psi\check{\psi} = \hat{\psi}\psi$$

$$(I \vee \psi) \circ (I \vee \psi) = I \vee \psi \vee \psi \circ (I \vee \psi) = I \vee \psi$$

Thus $I \vee \psi$ is an extensive filter, i.e. a closing. But $\psi < \hat{\psi}$ implies $I \vee \psi < \hat{\psi}$. Hence $\hat{\psi} = I \vee \psi$. It follows

$$\psi \hat{\psi} = \psi \circ (I \vee \psi) = \psi$$

because ψ is a $\vee$-under filter.

Now, if ψ is an underfilter or a $\vee$-over filter, it is of the desired form $\psi = \varphi g$ or $\psi = g\varphi$ with $\varphi = \hat{\psi} > g = \psi$. Conversely, $\varphi > g$ implies $\varphi\varphi = \varphi > \varphi g$ and thus $\varphi g\ \varphi g < \varphi\varphi g = \varphi g$, and φg is an under filter. In the same way $g\varphi$ is a $\vee$-under filter by the inequalities

$$g\varphi < g\varphi \circ (I \vee g\varphi) < g \circ \varphi \circ (I \vee \varphi) = g\varphi$$

THEOREM 4-11 - For any $\psi \in \mathcal{T}'$, the under filtering closing $F\psi$, the $\vee$-under filtering closing $F_{\vee}\psi$, the over filtering opening $G\psi$ and the $\wedge$-over filtering opening $G_{\wedge}\psi$ are given by the relations :

$$F\psi = \hat{\psi}\psi \quad ; \quad F_{\vee}\psi = \psi\hat{\psi}$$

$$G\psi = \check{\psi}\psi \quad ; \quad G_{\wedge}\psi = \psi\check{\psi}$$

Let g be an under filtering, so that $g = \hat{g}g$ by Criterion 4-8. If $g > \psi$, we also have $\hat{g} > \hat{\psi}$, and thus $g = \hat{g}g > \hat{\psi}\psi > \psi$. It follows from Criterion 4-8 that $\hat{\psi}\psi$ is the smallest under filter $> \psi$, i.e. $\hat{\psi}\psi = F\psi$. The proof is about the same for the three other relations.

COROLLARY 1 - An increasing mapping $\psi \in \mathcal{T}'$ is a filter if and only if $\hat{\psi}\psi = \check{\psi}\psi$. ψ is a strong filter if and only if $\psi\hat{\psi} = \psi\check{\psi}$. ψ is a $\vee$-filter if and only if $\check{\psi}\psi = \psi\hat{\psi}$ and ψ is a $\wedge$-filter if and only if $\psi\check{\psi} = \hat{\psi}\psi$.

COROLLARY 2 - Let ψ be a filter and let $\tilde{I}$ and $\underset{\sim}{I}$ denote the closing and the opening associated with $\tilde{\mathcal{B}}_{\psi}$ and $\underset{\sim}{\mathcal{B}}_{\psi}$. Then we have :

$$F_{\vee}\psi = \psi\hat{\psi} = \check{\psi}\hat{\psi} = \underset{\sim}{I}\hat{\psi} \quad ; \quad G_{\wedge}\psi = \psi\check{\psi} = \hat{\psi}\check{\psi} = \underset{\sim}{I}\check{\psi}$$

In particular, ψ is a $\vee$-filter (resp. a $\wedge$-filter) if and only if it is of the form $\psi = \gamma\varphi$ (resp. $\psi = \varphi\gamma$) for an opening γ and a closing φ.

Proof - We have $\underset{\sim}{I} < \check{\psi} < \psi$, and thus

$$\underset{\sim}{I}\,\hat{\psi} < \check{\psi}\,\hat{\psi} < \psi\,\hat{\psi}$$

But, by relation (4.5), $\underset{\sim}{I}\psi = \psi$, and thus $\psi\hat{\psi} = \underset{\sim}{I}\psi\hat{\psi} < \underset{\sim}{I}\check{\psi}\hat{\psi} = \underset{\sim}{I}\hat{\psi}$. $\underset{\sim}{I}\hat{\psi} = \check{\psi}\hat{\psi} = \psi\hat{\psi}$ follow.

Now, if ψ is a $\vee$-filter, we have $\psi = \psi\hat{\psi} = \check{\psi}\hat{\psi}$, so that ψ is of the desired form $\gamma\varphi$ with $\gamma = \check{\psi}$ and $\varphi\varphi = \hat{\psi}$. Conversely, $\gamma\varphi$ is a $\vee$-filter by Criterion 4-7.

4.4.1 - Invariance of the Invariance domain.

THEOREM 4-12 - Let ψ be an increasing mapping. Then ψ and its four envelopes $\psi\hat{\psi}$, $\hat{\psi}\psi$, $\psi\check{\psi}$ and $\check{\psi}\psi$ have the same invariance domain. Moreover, ψ, $\psi\hat{\psi}$ and $\hat{\psi}\psi$ have the same anti-extensivity domain $\hat{\mathcal{B}}_{\psi}$, on which they are equal. In the same way, ψ, $\psi\check{\psi}$ and $\check{\psi}\psi$ have the same extensivity domain $\check{\mathcal{B}}_{\psi}$ on which they are equal.

Proof - By $\hat{\psi} > \psi$, we have $\hat{\psi} = \hat{\psi}\hat{\psi} > \psi\hat{\psi}$. But $F_{\vee} > F$ implies $F_{\vee}\psi = \psi\hat{\psi} > F\psi = \hat{\psi}\psi > \psi$, so that :

$$\hat{\psi} > \psi\hat{\psi} > \hat{\psi}\psi > \psi$$

Hence $\hat{\psi}$ is the smallest closing $> \psi\hat{\psi}$ and also the smallest closing $> \hat{\psi}\psi$. Thus, by Theorem 4-10, ψ, $\psi\hat{\psi}$ and $\hat{\psi}\psi$ have the same anti-extensivity domain $\hat{\mathcal{B}}_{\psi}$. But $\psi\hat{\psi} = \psi$ on $\hat{\mathcal{B}}_{\psi}$, because $\hat{\psi} = I$ on $\hat{\mathcal{B}}_{\psi}$, and also $\hat{\psi}\psi = \psi$ on $\hat{\mathcal{B}}_{\psi}$, because $\psi\hat{\psi} > \hat{\psi}\psi > \psi$. Then, for any $A \in \mathcal{P}$, $\hat{\psi}\psi(A) = A$ implies $A \in \hat{\mathcal{B}}_{\psi}$, and thus $A = \hat{\psi}\psi(A) = \psi(A) \in \mathcal{B}_{\psi}$. In the same way, $\psi\hat{\psi}(A) = \hat{\psi}\psi\hat{\psi}(A)$

because $\phi = \hat{\phi}\phi$ on $\hat{\mathcal{B}}_\phi$, and thus $\phi\hat{\phi}(A) \in \hat{\mathcal{B}}_\phi$. Then $\phi\hat{\phi}(A) = A$ again implies $A \in \hat{\mathcal{B}}_\phi$ and $A = \phi\hat{\phi}(A) = \phi(A) \in \mathcal{B}_\phi$. Conversely, if $A \in \mathcal{B}_\phi$, we have $A \in \hat{\mathcal{B}}_\phi$, because $\mathcal{B}_\phi \subset \hat{\mathcal{B}}_\phi$, and thus $A = \phi(A) = \hat{\phi}\phi(A) = \phi\hat{\phi}(A)$.

<u>COROLLARY</u> - $\mathcal{B}_\phi$ is a complete lattice. The two filters $GF\phi$ and $FG\phi$, the three $\wedge$-filters $G_\wedge F\phi$, $F G_\wedge \phi$ and $G_\wedge F_\vee \phi$ and the three $\vee$-filters $G F_\vee \phi$, $F_\vee G \phi$ and $F_\vee G_\wedge \phi$ belong to $\mathfrak{J}d(\mathcal{B}_\phi)$.

Note that if the lattice $\mathcal{P}$ is modular, $G_\wedge F_\vee \phi$ and $F_\vee G_\wedge \phi$ are strong filters : in this case, for any $\mathcal{B} \subset \mathcal{P}$ such that $\mathfrak{J}d(\mathcal{B})$ is not empty, there are strong filters in $\mathfrak{J}d(\mathcal{B})$.

<u>THEOREM 4-13</u> - For any increasing mapping Ψ we have the equalities

$$G F_\vee \phi = \underset{\sim}{I}\,\hat{\phi} = \check{\phi}\,\hat{\phi} \quad ; \quad GF\phi = \underset{\sim}{I}\,\hat{\phi}\,\phi = \check{\phi}\,\hat{\phi}\,\phi$$

$$F G_\wedge \phi = \tilde{I}\,\check{\phi} = \hat{\phi}\,\check{\phi} \quad ; \quad FG\phi = \tilde{I}\,\check{\phi}\,\phi = \hat{\phi}\,\check{\phi}\,\phi$$

where $\underset{\sim}{I}$ and $\tilde{I}$ are the opening and the closing associated with $\underset{\sim}{\mathcal{B}}_\phi$ and $\tilde{\mathcal{B}}_\phi$.

<u>Proof</u> - $\check{\phi}\hat{\phi}$ is a $\vee$-filter (Criterion 4-7), so that $\check{\phi}\hat{\phi} < F_\vee\phi = \phi\hat{\phi}$ implies $\check{\phi}\hat{\phi} < G F_\vee \phi$. Moreover, $\underset{\sim}{I} < \check{\phi}$, so that we may write :

$$\underset{\sim}{I}\,\hat{\phi} < \check{\phi}\,\hat{\phi} < G F_\vee \phi < F_\vee \phi = \phi\,\hat{\phi} < \hat{\phi}$$

When multiplying by $\underset{\sim}{I}$ it follows $\underset{\sim}{I}\,\hat{\phi} < \underset{\sim}{I} G F_\vee \phi < \underset{\sim}{I}\,\hat{\phi}$, and thus $\underset{\sim}{I}\,\hat{\phi} = \underset{\sim}{I}\, G F_\vee \phi$. But $G F_\vee \phi$ is a filter, and its invariance domain is $\mathcal{B}_\phi$ (Theorem 4-12, corollary). Thus, by relation (4.5), $\underset{\sim}{I}\, G F_\vee \phi = G F_\vee \phi$. It follows $\underset{\sim}{I}\,\hat{\phi} = G F_\vee \phi$, and, by the inequalities $\underset{\sim}{I}\,\hat{\phi} < \check{\phi}\,\hat{\phi} < G F_\vee \phi$, we also have $\check{\phi}\,\hat{\phi} = G F_\vee \phi = \underset{\sim}{I}\,\hat{\phi}$.

Note that $\phi\check{\phi}\hat{\phi} = \check{\phi}\hat{\phi}$, because the invariance domain of the filter $\check{\phi}\hat{\phi}$ is $\mathcal{B}_\phi$ and thus, by Criterion 4-3, $\underset{\sim}{I}\,\hat{\phi}\,\phi = \check{\phi}\hat{\phi}\phi$ is a filter. Then, the inequality $\underset{\sim}{I}\,\hat{\phi}\,\phi < \hat{\phi}\,\phi = F\phi$

implies $\underset{\sim}{I}\ \hat{\phi}\ \phi < G\ F\ \phi$. On the other hand, $G\ F\ \phi \in \mathcal{J}d(\mathcal{B}_\phi)$ by Theorem 4-12, and thus $\underset{\sim}{I}\ G\ F\ \phi = G\ F\ \phi$ by relations(4.5). Hence $G\ F\ \phi = \underset{\sim}{I}\ G\ F\ \phi < \underset{\sim}{I}\ F\ \phi = \underset{\sim}{I}\ \hat{\phi}\ \phi$ and the equality $\underset{\sim}{I}\ \hat{\phi}\ \phi = G\ F\ \phi$.

THEOREM 4-14 - Let f_i be a family of over filters (resp. under filters) which have the same invariance domain $\mathcal{B}$. Then $\mathcal{B}$ is the invariance domain of $\vee f_i$ (resp. $\wedge f_i$).

Proof - Let for instance f_i be a family of over filters. Put $f = \vee f_i$. Then f is an over filter (Theorem 4-3)n Ff and F f_i are filters (Theorem 4-7). From the formula $F(\vee f_i) = F(\vee F f_i)$ which is true for any closing, we have $Ff = F(\vee F f_i)$. But $F f_i \in \mathcal{J}d(\mathcal{B})$, by Theorem 4-11. Thus $Ff = F(\vee F f_i) \in \mathcal{J}d(\mathcal{B})$ (Theorem 4-8). Now f and F f have the same invariance domain, by Theorem 4-12, and we conclude $\mathcal{B}_f = \mathcal{B}$.

For any $\phi \in \mathcal{P}'$, ϕ, $\phi\ o(I \vee \phi)$, $(I \vee \phi)\ o\ \phi$, $\phi\ o(I \wedge \phi)$ and $(I \wedge \phi)\ o\ \phi$ have the same invariance domain $\mathcal{B}$.

Proof - The inequalities $\phi \subset \phi\ o\ (I \vee \phi) \subset \phi\ \hat{\phi} = F_\vee \phi$ imply $F_\vee(\phi\ o(I \vee \phi)) = F_\vee \phi$. Thus, by Theorem 4-12, ϕ and $\phi\ o(I \vee \phi)$ have the same invariance domain. The proof is the same in the other cases.

In the same way, if g is an over filter or an under filter, g and gg have the same invariance domain. For $gg = (I \vee g)\ o\ g$ if g is an over filter, and $gg = (I \wedge g)\ o\ g$ if g is an under filter.

The following theorem is only a summary :

THEOREM 4-15 - The class $\mathcal{P}'(\mathcal{B})$ of the increasing mappings which have the same invariance domain $\mathcal{B}$ is closed under the two over and the two under compositions. It is also closed under the four envelopes F, $F_\vee$, G, $G_\wedge$. The subclass of the over filters (resp. under filters) of $\mathcal{P}'(\mathcal{B})$ is closed under these eight operations, and also under $\vee$ (resp. $\wedge$) and self composition. Moreover, if the lattice $\mathcal{P}$ is modular, the subclass of the $\wedge$-over filters (resp. $\vee$-under filters) is closed under these ten operations.

4.5 - THE MIDDLE ELEMENT.

Throughout this section, the lattice $\mathcal{P}$ is assumed to be modular, so that Theorem 4-9 may be applied. Let f be a $\vee$-under filter, and let g be a $\wedge$-over filter such that $f > g$. Let $f' = G_\wedge f$ and $g' = F_\vee g$, so that f' and g' are strong filters (Theorem 4-9). Note that $g < f$ implies $g < f' = G_\wedge f$, because g is a $\wedge$-over filter and thus $g' = F_\vee g < f'$, because f' is a strong filter, so that we have :

$$f > f' > g' > g$$

Now $f > f' = f\check{f}$ (Theorem 4-11) and $f\check{f} > \check{f}$ imply that $\check{f}$ is the greatest opening $< f'$, i.e. $\check{f} = \check{f}'$. But $\check{f} = I \wedge f'$, because f' is a $\wedge$-filter. In the same way, $\hat{g}' = I \vee g' = \hat{g}$. Then, let us define a new element $\alpha \in \mathcal{P}'$ by writing

$$\alpha = \check{f}\,\hat{g} = (I \wedge f') \circ (I \vee g')$$

Note that we have $f' < f' \circ (I \vee g') < f' \circ (I \vee f') = f'$, because f' is a strong filter, and thus $f' \circ (I \vee g') = f'$. Hence :

$$\alpha = (I \wedge f') \circ (I \vee g') = (I \vee g') \wedge f'$$

In the same way, starting from

$$\alpha' = \hat{g}\,\check{f} = (I \vee g') \circ (I \wedge f')$$

we find $\alpha' = (I \wedge f') \vee g'$. But the lattice $\mathcal{P}$ is modular, so that $(I \vee g') \wedge f' = (I \wedge f') \vee g'$, i.e. $\alpha = \alpha'$, and we may write :

$$\alpha = \check{f}\,\hat{g} = \hat{g}\,\check{f}$$

Thus α is a <u>strong filter</u>. We have $\alpha < f\hat{f} = f$, and thus $\alpha < f' = G \wedge f$, because α is a strong filter. In the same way we find $\alpha > g'$, and finally

$$g < g' < \alpha < f' < f$$

Note that $\alpha = (I \vee g') \wedge f$ is equal to f' if and only if $\hat{g} = I \vee g' > f$, i.e. $\hat{g} = I \vee f'$. In the same way $\alpha = f$ if and only if $\hat{g} = \hat{f}$, and in this case f is a strong filter.

Note also that the inequalities $\check{f} < \alpha = \check{f}\,\hat{g} < f\,\hat{f} = f$ imply $\check{\alpha} = \check{f}$, and in the same way $\hat{\alpha} = \hat{g}$. Now the invariance domain of α is $\mathcal{B}_\alpha = \hat{\mathcal{B}}_\alpha \cap \check{\mathcal{B}}_\alpha$, and then we have

$$\mathcal{B}_\alpha = \hat{\mathcal{B}}_g \cap \check{\mathcal{B}}_f$$

Let us summarize these results.

<u>THEOREM 4-16</u> - If f is a $\vee$-under filter and g is a $\wedge$-over filter on a modular lattice $\mathcal{P}$ and $f > g$, there exists a strong filter α, called the middle element of f and g, such that

$$\check{\alpha} = \check{f} \quad ; \quad \hat{\alpha} = \hat{g}$$

$$\alpha = \check{f}\,\hat{g} = \hat{g}\,\check{f} = \hat{g} \wedge f\check{f} = \check{f} \vee g\hat{g}$$

$$g < g\hat{g} < \alpha < f\check{f} < f$$

$$\mathcal{B}_\alpha = \hat{\mathcal{B}}_g \cap \check{\mathcal{B}}_f$$

Moreover, $\alpha = f$ (resp. $\alpha = g$) if and only if $\hat{g} = \hat{f}$ (resp. $\check{f} = \check{g}$).

With the same notation~~s~~, if ϕ is a strong filter such that

$$g < \phi < f$$

we find $\phi\alpha = \hat{\phi}\check{\phi}\,\check{f}\hat{g} = \hat{\phi}\check{\phi}\hat{g}$, because $\phi < f$ implies $\check{\phi} < \check{f}$ and $\check{\phi}\check{f} = \check{\phi}$. But $\hat{\phi}\check{\phi} = \check{\phi}\hat{\phi}$, because ϕ is a strong filter, and $\hat{\phi}\,\hat{g} = \hat{\phi}$, because $\phi > g$ implies $\hat{\phi} > \hat{g}$ and $\hat{\phi}\hat{g} = \hat{\phi}$. Thus $\phi\alpha = \hat{\phi}\check{\phi}\hat{g} = \check{\phi}\hat{\phi}\hat{g} = \check{\phi}\hat{\phi} = \phi$. In the same way, we find $\alpha\,\phi = \phi$. In other words :

$$\phi\, \alpha = \alpha\, \phi = \phi$$

Since $\alpha\,\phi = \phi$, by Criterion 4-2 we have $\phi \in \mathcal{J}d(\mathcal{B}_\alpha)$ if and only if $\phi\,\alpha = \alpha$, i.e. $\alpha = \phi$ because $\phi\,\alpha = \phi$. Thus :

COROLLARY 1 - The middle element α is the only strong filter $\phi \in \mathcal{J}d(\mathcal{B}_\alpha)$ such that $g < \phi < f$.

If f and g admit the same invariance set $\mathcal{B}$, we have $\mathcal{B} = \hat{\mathcal{B}}_g \cap \check{\mathcal{B}}_g \subset \hat{\mathcal{B}}_g \cap \check{\mathcal{B}}_f$, because $g < f$ implies $\check{g} < \check{f}$ and $\check{\mathcal{B}}_g \subset \hat{\mathcal{B}}_f$. But $\hat{\mathcal{B}}_g \cap \check{\mathcal{B}}_f = \mathcal{B}_\alpha$ by Theorem 4-16, so that $\mathcal{B} \subset \mathcal{B}_\alpha$ and thus $\tilde{I}_{\mathcal{B}} > \tilde{I}_{\mathcal{B}_\alpha}$. Now the greatest element of $\mathcal{J}d(\mathcal{B}_\alpha)$ is $\alpha_M = \alpha\, \tilde{I}_{\mathcal{B}_\alpha}$, by relation (4.5). Thus :

$$\alpha_M < \alpha\, \tilde{I}_{\mathcal{B}} = \check{f}\, \hat{g}\, \tilde{I}_{\mathcal{B}} = \check{f}\, \tilde{I}_{\mathcal{B}} = \check{f}\, \hat{f}\, \tilde{I}_{\mathcal{B}}$$

But $\check{f}\,\hat{f} \in \mathcal{J}d(\mathcal{B})$ (Theorem 4-12 and 4-13) and thus, by relation (4.5), $\check{f}\hat{f}\,\tilde{I}_{\mathcal{B}} = \underset{\sim}{I}_{\mathcal{B}}\tilde{I}_{\mathcal{B}} = \phi_M$ is the greatest element of $\mathcal{J}d(\mathcal{B})$. In the same way, we find that the smallest element α_m of $\mathcal{J}d(\mathcal{B}_\alpha)$ is greater than the smallest element ϕ_m of $\mathcal{J}d(\mathcal{B})$:

$$\phi_m < \alpha_m < \alpha < \alpha_M < \phi_M$$

In the particular case $f = \phi_M$, $g = \phi_m$, it follows that any $\phi \in \mathcal{J}d(\mathcal{B}_\alpha)$ is $> \alpha_m > g = \phi_m$ and $< f = \phi_M$. Thus, by corollary 1 , α is the only strong filter in $\mathcal{J}d(\mathcal{B}_\alpha)$. In this case, $\alpha \in \mathcal{J}d(\mathcal{B})$ if and only if there is only one strong filter in $\mathcal{J}d(\mathcal{B})$: we have just seen that this condition is necessary. Conversely, $\phi_m < \alpha < \phi_M$ implies $F_\vee\, \phi_m < \alpha < G_\wedge\, \phi_M$, because α is a strong filter. But the two strong filters $F_\vee\, \phi_m$ and $G_\wedge\, \phi_M$ belong to $\mathcal{J}d(\mathcal{B})$ (Theorem 4-12) and thus are equal if $\mathcal{J}d(\mathcal{B})$ contains only one strong filter. But this implies $\alpha = F_\vee\, \phi_m = G_\wedge\, \phi_M \in \mathcal{J}d(\mathcal{B})$. Thus :

COROLLARY 2 - If f and g are the greatest and the smallest element of a same class $\mathcal{J}d(\mathcal{B})$, there is no other strong filter in $\mathcal{J}d(\mathcal{B}_\alpha)$ than the middle element α itself. Moreover, we have $\alpha \in \mathcal{J}d(\mathcal{B})$, i.e. $\mathcal{B} = \mathcal{B}_\alpha$ if and only if $\mathcal{J}d(\mathcal{B})$ contains only one strong filter.

<u>EXAMPLE</u> : If γ is an opening and φ is a closing, we may choose

$$f = \varphi\,\gamma\,\varphi \quad ; \quad g = \gamma\,\varphi\,\gamma$$

(see Criteria 4-6 and 4-7, and the example). If f and g are strong filters, the middle element α is of the very simple form :

$$\alpha = (I \vee \gamma\varphi\gamma) \wedge \varphi\gamma\varphi = (I \wedge \varphi\gamma\varphi) \vee \gamma\varphi\gamma$$

For instance, suppose that E is a <u>topological space</u> and $\mathcal{P} = \mathcal{P}(E)$ is the lattice of the subsets $A \subset E$. For any $A \in \mathcal{P}$, $\mathring{A}$ is the interior of A, and $\overline{A}$ is the adherence of A. Let γ and φ be defined by :

$$\gamma(A) = \mathring{A} \quad ; \quad \varphi(A) = \overline{A}$$

Then $\varphi\gamma\varphi$ and $\gamma\varphi\gamma$ are <u>strong filters</u> , and the middle element α between $\varphi\gamma\varphi$ and $\gamma\varphi\gamma$ is defined by :

$$\alpha(A) = (A \cup \mathring{\overline{\mathring{A}}}) \cap \overline{\mathring{\overline{A}}} = (A \cap \overline{\mathring{\overline{A}}}) \cup \mathring{\overline{\mathring{A}}}$$

<u>Proof</u> - By Criterion 4-7 and the example which follows, it is sufficient to prove that $\varphi\gamma$ is a $\vee$-filter. By duality, this implies that $\gamma\varphi$ is a $\wedge$-filter, and thus $\varphi\gamma\varphi$ and $\gamma\varphi\gamma$ are strong filters.

Let A be a subset of E. Put $A' = \overline{\mathring{A}} \cup A$. We must prove the equality :

$$\overline{\mathring{A}} = \overline{\mathring{A'}}$$

But two closed sets are equal if and only if they meet the same open sets. For any open set G we have

$$G \cap \overline{\mathring{A}} \neq \emptyset$$

if and only if there is another open set $G' \subset G$ such that

..

$G' \subset A$. In the same way, we have $G \cap \overline{\mathring{A}'} \neq \emptyset$ if and only if there is an open set $G'' \subset G$ such that

$$G'' \subset A' = A \cup \overline{\mathring{A}} .$$

But $G'' \subset A \cup \overline{\mathring{A}}$ implies either $G'' \subset A$ or $G'' \cap \overline{\mathring{A}} \neq \emptyset$, so that in both cases we have

$$G \cap \overline{\mathring{A}} \neq \emptyset .$$

Conversely, $G \cap \overline{\mathring{A}} \neq \emptyset$ implies $G \cap \overline{\mathring{A}'} \neq \emptyset$, because $A' \supset \overline{\mathring{A}}$. Being met by the same open sets, the two closed sets $\overline{\mathring{A}'}$ and $\overline{\mathring{A}}$ are equal, and $\varphi\gamma$ is a strong filter .

4.6 - FILTERS ON $\mathcal{F}(E)$ or $\tilde{\mathcal{K}}(E)$.

Let E be a locally compact and Hausdorff topological space. Then, the space $\mathcal{F}(E)$ of the closed subsets of E is a complete lattice with respect to the order $\subset$: the infimum is the usual intersection $\cap F_i$, but the supremum is the closed union $\overline{\cup F_i}$ and not the ordinary union $\cup F_i$. In the same way, the space $\mathcal{G}(E)$ of the open subsets of E is a complete lattice, with $\vee G_i = \cup G_i$ and $\wedge G_i = \mathring{\cap} (G_i)$, i.e. interior of $\cap G_i$.

Concerning the space $\mathcal{K}(E)$ of the compact subsets of E, it is not a complete lattice, except if the space E itself is compact, because it does not contain a greatest element. For this reason, we add the element E and write :

$$\tilde{\mathcal{K}} = \mathcal{K} \cup \{E\}$$

This notation is consistent. Strictly speaking, the space $\mathcal{K}$ is not closed under intersection, although it contains the intersection $\cap K_i$ for any non empty family of compact sets K_i, since it does not contain the intersection of the empty family, which is E itself. For this reason, the space $\mathcal{K} \cup \{E\}$ is really the class closed under intersection spanned by $\mathcal{K}$, i.e. $\mathcal{K} \cup \{E\} = \tilde{\mathcal{K}}$.

In $\tilde{\mathcal{K}}$, the supremum and the infimum are given by the formulae

$$\bigwedge_{i\in J} K_i = \bigcap K_i \quad \text{if} \quad J \neq \emptyset \quad , \quad \bigwedge_{i\in \emptyset} K_i = E$$

$\vee K_i = \overline{\bigcup K_i}$ if $\overline{\bigcup K_i} \in \mathcal{K}$, and $\vee K_i = E$ otherwise, so that $\tilde{\mathcal{K}}$ is a complete lattice.

From the topological point of view, the space $\tilde{\mathcal{K}}$ may be identified with the <u>compactification</u> of the locally compact space $\mathcal{K}$ provided with the myope topology. The point E is the point at infinity, i.e. its open neighbourhoods are the complements of the compact subsets of $\mathcal{K}$. In particular, the family of the subsets of $\mathcal{K}$ of the form $\{K : K \in \mathcal{K}, K \not\subset K_0\}$, $K_0 \in \mathcal{K}$ constitutes a fundamental system of open neighborhoods of the point $E \in \tilde{\mathcal{K}}$. A sequence K_n in $\tilde{\mathcal{K}}$ converges towards E if it has no accumulation point in $\mathcal{K}$, i.e. if for any $K_0 \in \mathcal{K}$ we have $K_n \not\subset K_0$ for n great enough. See [1].

If ψ is <u>a filter</u> on the complete lattice $\tilde{\mathcal{K}}$, its invariance domain $\mathcal{B} \subset \tilde{\mathcal{K}}$ is also a complete lattice (Theorem 4-2). Its anti-extensivity domain $\hat{\mathcal{B}}$ and the class $\tilde{\mathcal{B}}$ are closed under $\cap$. Its extensivity domain $\check{\mathcal{B}}$ and the class $\underset{\sim}{\mathcal{B}}$ are closed under $\vee$, but not under the ordinary union $\cup$.

We recall that an increasing mapping ψ from $\tilde{\mathcal{K}}$ into itself is <u>upper semi-continuous</u> (<u>u.s.c.</u>) if for any set $K \in \tilde{\mathcal{K}}$ and any open set $G \in \mathcal{G}$ such that $G \supset \psi(K)$ there is an open set $G' \supset K$ such that $K' \subset G'$ implies $\psi(K') \subset G$ for any $K' \in \tilde{\mathcal{K}}$.

If <u>a filter</u> ψ on $\tilde{\mathcal{K}}$ <u>is u.s.c.</u>, its <u>extensivity domain $\check{\mathcal{B}}$ is closed</u> in $\tilde{\mathcal{K}}$.

<u>Proof</u> - If $K \in \tilde{\mathcal{K}}$ and $K \not\subset \psi(K)$, there exist two open sets G_0 , G_1 such that $G_0 \supset \psi(K)$, $K \cap G_1 \neq \emptyset$, $G_0 \cap G_1 = \emptyset$. If ψ is u.s.c., there also exists an open set $G' \supset K$ such that $K' \subset G'$ implies $\psi(K') \subset G_0$. Thus the open neighborhood $\mathcal{O}$ of K defined by

$$\mathcal{O} = \{K' : K' \subset G' , K' \cap G_1 \neq \emptyset\}$$

is disjoint from $\check{\mathcal{B}}$: thus $\check{\mathcal{B}}$ is closed in $\mathcal{K}$.

We know that $\check{\mathcal{B}}$ is closed in $\mathcal{K}$ if and only if the corresponding opening $\check{\phi}$ is u.s.c. (see [1]). In other words : if a filter ϕ on $\mathcal{K}$ is u.s.c. , the greatest opening $\check{\phi} \subset \phi$ is also u.s.c.

A v-filter on $\mathcal{K}$ is u.s.c. if and only if $\check{\phi}$ and $\hat{\phi}$ are u.s.c.

Proof - If a v-filter ϕ is u.s.c., $\check{\phi}$ is u.s.c. (see above). But $\hat{\phi} = I \cup \phi$, because ϕ is a v-filter, and $I \cup \phi$ is u.s.c., as is the case for any finite union of u.s.c. increasing mappings. Conversely, if $\check{\phi}$ and $\hat{\phi}$ are u.s.c., $\phi = \check{\phi}\,\hat{\phi}$ is u.s.c., because the composition of two increasing u.s.c. mappings is u.s.c.

The class of the increasing u.s.c. mappings from $\mathcal{K}$ into itself is closed under $\cap$, so that any increasing mapping ϕ has a u.s.c. closing ϕ_k which is the smallest u.s.c. mapping greater than ϕ. Its explicit construction is as follows : first put for any $G \in \mathcal{G}$:

$$\phi_g(G) = \cup \{\phi(K), K \in \mathcal{K} , K \subset G\}$$

Note that in general $\phi_g(G) \notin \mathcal{G}$. Then, for any $K \in \mathcal{K}$:

$$\phi_k(K) = \cap \{\phi_g(G), G \in \mathcal{G}, G \supset K\}$$

Then ϕ_k is the u.s.c. closing of ϕ and may be rewritten in a more direct form :

$$\phi_k(K) = \cap \{\phi(K'), K' \in \mathcal{K} , \mathring{K}' \supset K\} \tag{4.7}$$

Proof - For any $K \in \mathcal{K}$, $G \in \mathcal{G}$, such that $K \subset G$, there is another compact set $K' \in \mathcal{K}$ such that $K \subset \mathring{K}' \subset K' \subset G$. It follows

$\psi_k(K) \subset \psi_g(\mathring{K}') \subset \psi(K') \subset \psi_g(G)$ and the equality (4.7). By this equality, ψ_k maps $\tilde{\mathcal{K}}$ into itself.

Let G be an open set such that $G \supset \psi_k(K)$. If $K = E$, we have $\psi_k(K') \subset \psi_k(E)$ for any $K' \subset E$. If $K \neq E$, i.e. $\mathring{K} \in \mathcal{K}$, the class of the compact sets $G^c \cap \psi(K')$, $K' \in \mathcal{K}$, $\mathring{K}' \supset K$ has an empty intersection. Thus, there exist finite subclasses K_i', $i = 1,\dots,n$ such that $\mathring{K}_i' \supset K$ and $\cap\, \psi(K_i') \subset G$. Let $K' = \cap K_i'$. We have $\mathring{K}' \supset \cap \mathring{K}_i' \supset K$ and $\psi(K') \subset \cap\, \psi(K_i') \subset G$. In other words there exists a compact set $K' \in \mathcal{K}$ such that $\mathring{K}' \supset K$ and $\psi(K') \subset G$. Now, for any other compact set $K'' \subset \mathring{K}'$, we have $\psi_k(K'') \subset \psi(K') \subset G$. Thus ψ_k is u.s.c.

Now, if ψ' is u.s.c. and $\psi' \supset \psi$, we find $\psi_g' \supset \psi_g$ and $\psi_k' \supset \psi_k$. But it is easy to see that $\psi_k' = \psi'$ for any increasing u.s.c. mapping. Then $\psi' \supset \psi_k$, and ψ_k is the u.s.c. closing of ψ.

If ψ is a u.s.c. filter, its <u>invariance domain</u> $\mathcal{B}$ is closed under decreasing intersection : if $K_n = \psi(K_n)$ and $K_n \downarrow K$, we have $\psi(K_n) \downarrow \psi(K)$, because ψ is u.s.c. and thus $K = \psi(K)$.

The notation $\mathcal{J}d(\mathcal{B},\tilde{\mathcal{K}})$ will denote the class of the filters on $\tilde{\mathcal{K}}$ which have the same invariance domain $\mathcal{B} \subset \tilde{\mathcal{K}}$.

<u>THEOREM 4-17</u> - Let $\mathcal{B}$ be a subset of $\tilde{\mathcal{K}}$ such that $\mathcal{J}d(\mathcal{B},\tilde{\mathcal{K}}) \neq \emptyset$. Then, for any $\psi \in \mathcal{J}d(\mathcal{B},\tilde{\mathcal{K}})$, the u.s.c. closing ψ_k is a filter and belongs to $\mathcal{J}d(\mathcal{B},\tilde{\mathcal{K}})$ if and only if the greatest element ψ_M of $\mathcal{J}d(\mathcal{B},\tilde{\mathcal{K}})$ is u.s.c.

<u>Proof</u> - If $\psi_k \in \mathcal{J}d(\mathcal{B},\tilde{\mathcal{K}})$ for any $\psi \in \mathcal{J}d(\mathcal{B},\tilde{\mathcal{K}})$, we have $(\psi_M)_k \in \mathcal{J}d(\mathcal{B},\tilde{\mathcal{K}})$ and $(\psi_M)_k \supset \psi_M$. Thus, $\psi_M = (\psi_M)_k$, because ψ_M is the greatest element of $\mathcal{J}d(\mathcal{B},\tilde{\mathcal{K}})$, i.e. ψ_M is u.s.c.

Conversely, for any increasing mappings ψ_o, ψ, if ψ_o is u.s.c. , we have :

$$\phi_o\, \phi_k(K) = \phi_o(\cap \{\phi(K'), \mathring{K}' \supset K) = \cap \{\phi_o\, \phi(K'), \mathring{K}' \supset K\}$$

If ϕ_o and ϕ have the same invariance domain, we have $\phi_o\, \phi = \phi$. Hence :

$$\phi_o\, \phi_k = \phi_k$$

Concerning $\phi_k\, \phi_o$, we only find $\phi_k\, \phi_o \supset \phi\, \phi_o = \phi_o$. But, if $\phi_o \supset \phi$, we also have $\phi_o \supset \phi_k$, because ϕ_o is u.s.c., and thus $\phi_o = \phi_o\, \phi_o \supset \phi_k\, \phi_o \supset \phi_o$. It follows :

$$\phi_k\, \phi_o = \phi_o$$

and $\phi_k \in \mathcal{J}d(\phi_o, \mathcal{K})$ by Criterion 4-2. In particular, if the maximal element of $\mathcal{J}d(\mathcal{B}, \mathcal{K})$ is u.s.c., it follows that $\phi_k \in \mathcal{J}d(\mathcal{B}, \mathcal{K})$ for any $\phi \in \mathcal{J}d(\mathcal{B}, \mathcal{K})$.

4.6.1 - Open Filters.

In general, if ϕ is <u>a filter</u> on $\mathcal{K}$, its <u>u.s.c. closing</u> ϕ_k is not a filter, but only an <u>over filter</u>.

<u>Proof</u> - $\phi_k \supset \phi$ implies $\phi_k\, \phi_k \supset \phi\, \phi = \phi$. But $\phi_k\, \phi_k$ is u.s.c. since it is the composition of two u.s.c. increasing mappings. Thus $\phi_k\, \phi_k \supset \phi_k$. In the same way, the mapping ϕ_g from $\mathcal{G}$ into $\mathcal{P}(E)$ defined by

$$\phi_g(G) = \cup \{\phi(K),\ K \in \mathcal{K},\ K \subset G\} \quad (G \in \mathcal{G})$$

is <u>not</u> in general a mapping from $\mathcal{G}$ into $\mathcal{G}$. If it is, i.e. if $\phi_g(G) \in \mathcal{G}$ for any $G \in \mathcal{G}$, we say that the mapping ϕ is an <u>open mapping</u>. Then :

<u>THEOREM 4-18</u> - If a filter ϕ on $\mathcal{K}$ is open, then ϕ_g is a l.s.c. filter on $\mathcal{G}$, and ϕ_k a u.s.c. filter on $\mathcal{K}$.

<u>Proof</u> - In a more general context, if $\mathcal{A}$ is a complete lattice in $\mathcal{P}(E)$, and $\phi : \mathcal{A} \to \mathcal{A}$ a filter on $\mathcal{A}$, the smallest

extension $\underset{\sim}{\phi}$ and the greatest extension $\tilde{\phi}$ of ϕ on $\mathcal{P}(E)$ are defined by :

$$\begin{aligned} \underset{\sim}{\phi}(P) &= \bigcup \{\phi(A),\ A \in \mathcal{A},\ A \subset P\} \\ \tilde{\phi}(P) &= \bigcap \{\phi(A),\ A \in \mathcal{A},\ A \supset P\} \end{aligned} \qquad (P \in \mathcal{P}(E))$$

Then $\underset{\sim}{\phi}$ is <u>an over filter</u>, and $\tilde{\phi}$ is <u>an underfilter</u> :

$$\begin{aligned} \underset{\sim}{\phi}\,\underset{\sim}{\phi}(P) &= \underset{\sim}{\phi}\,(\bigcup \{\phi(A),\ A \in \mathcal{A},\ A \subset P\}) \\ &\supset \bigcup \{\underset{\sim}{\phi}\,\phi(A),\ A \in \mathcal{A},\ A \subset P\} = \underset{\sim}{\phi}(P) \end{aligned}$$

because $\underset{\sim}{\phi}\,\phi(A) = \phi\,\phi(A) = \phi(A)$ for any $A \in \mathcal{A}$.

In our case, $\phi_g(G) = \underset{\sim}{\phi}(G)$ for any $G \in \mathcal{G}$. If ϕ is open, it follows $\phi_g\,\phi_g(G) = \underset{\sim}{\phi}\,\underset{\sim}{\phi}(G) \supset \phi_g(G)$, and ϕ_g is an <u>over filter</u> on $\mathcal{G}$. But, for any compact set $K \subset G$, there is another compact set K' such that $K \subset \mathring{K}' \subset K' \subset G$, so that we have

$$\phi_g(G) = \bigcup \{\phi_g(\mathring{K}'),\ K' \subset G\}$$

If ϕ is open, $\phi_g(G)$ is the union of the <u>open</u> sets $\phi_g(\mathring{K}')$, $K' \subset G$, so that for any compact set $K \subset \phi_g(G)$ we can find $K' \subset G$ such that $K \subset \phi_g(\mathring{K}') \subset \phi(K') \subset \phi_g(G)$. But this implies $\phi(K) \subset \phi_g(G)$ for any $K \subset \phi_g(G)$ and thus :

$$\phi_g\,\phi_g(G) = \bigcup \{\phi(K),\ K \subset \phi_g(G)\} \subset \phi_g(G)$$

Thus ϕ_g is also an under filter on $\mathcal{G}$. Hence ϕ_g is a <u>filter</u> on $\mathcal{G}$. From the same inequalities $K \subset \phi_g(\mathring{K}') \subset \phi(K') \subset \phi_g(G)$ it follows that $\phi_g(G') \supset K$ for any $G' \in \mathcal{G}$ such that $G' \supset \phi(K')$, so that the filter ϕ_g is lower semi-continuous on $\mathcal{G}$.

Concerning ϕ_k, we know that it is an u.s.c. over filter. But, for any $K \in \mathcal{K}$, we have $\phi_k(K) = \tilde{\phi}_g(K)$, and the greatest extension $\tilde{\phi}_g$ of the filter ϕ_g on $\mathcal{G}$ is an under filter on $\mathcal{P}(E)$.

Thus $\psi_k\, \psi_k(K) = \tilde{\psi}_g\, \tilde{\psi}_g(K) \subset \tilde{\psi}_g(K) = \psi_k(K)$, and ψ_k is an under-filter on $\tilde{\mathcal{K}}$. Hence ψ_k is a u.s.c. filter on $\tilde{\mathcal{K}}$.

In the same way :

<u>THEOREM 4-19</u> - Let ψ be a filter on $\mathcal{G}$. Put

$$\begin{cases} \psi'_k(K) = \bigcap \{\psi(G),\ G \in \mathcal{G},\ G \supset K\} & (K \in \tilde{\mathcal{K}}) \\ \psi'_g(G) = \bigcup \{\psi'_k(K),\ K \in \tilde{\mathcal{K}},\ K \subset G\} & (G \in \mathcal{G}) \end{cases}$$

Then ψ'_g is a l.s.c. under filter on $\mathcal{G}$ and it is the greatest l.s.c. increasing mapping $\mathcal{G} \to \mathcal{G}$ smaller than ψ'. Moreover, if $\psi'_k(K) \in \tilde{\mathcal{K}}$ for any $K \in \tilde{\mathcal{K}}$, ψ'_g is an u.s.c. filter on $\tilde{\mathcal{K}}$ and ψ'_g is a l.s.c. filter on $\mathcal{G}$.

In the case $E = \mathbb{R}^n$, any increasing mapping $\psi = \tilde{\mathcal{K}} \to \tilde{\mathcal{K}}$ (resp. $\psi' : \mathcal{G} \to \mathcal{G}$) compatible with translations is an open mapping (resp. $\psi'_k(K) \in \tilde{\mathcal{K}}$ for any $K \in \tilde{\mathcal{K}}$) so that Theorem 4-18 (resp. 4-19) may be applied.