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RANDOM SETS THEORY, AND ITS APPLICATIONS TO STEREOLOGY

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ABSTRACT

In order to study objects forming a sub-set A of the euclidean space, mathematical morphology uses structuring figures B and notes the frequency of events such as "B hits A", "B is included into A" etc... Thus, a probabilistic formalism is associated with this experimental technique and facilitates its interpretation. If A is considered as a closed set, we obtain a random closed sets theory, closely connected with integral geometry. The functionals T defined by $T(K) = P(A \cap K \neq \emptyset)$ for K compact are characterized as alternating capacities of infinite order. Interesting classes of functionals T are obtained if A is indefinitely divisible or semi-markovian. At last, the mathematical notion of granulometry (size distribution) is studied by using an axiomatic method.

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0 - INTRODUCTION

Two mathematical theories are of great importance for Stereology : Integral Geometry, as it has been developed for instance by H. Hadwiger, [3], but also the probability theory, for stereology is generally concerned not with individuals, but rather with collections or populations of similar objects. This statistical point of view requires a specific probabilistic theory, which we may call a random sets theory, [6], [7]. It turns out that many results of integral geometry admit a probabilistic interpretation in terms of random sets, and conversely. My purpose in this paper is only to give a few examples of such connections between the two theories.

1 - STRUCTURAL ELEMENTS

In order to define the structure of an object relatively to a family $\mathcal{R}$ of possible relationships between elements or parts of this object, we must know for each $R \in \mathcal{R}$ if R is true or false. Let us consider the simple case of a medium with two components only. The first component constitutes a set A of space points, the second one the complementary set A^c . For instance, in the case of a porous medium, A is the union of the grains, while A^c is the union of the pores. Let us now denote by B a geometrical figure which will be used as a structuring tool. B is chosen among a given family $\mathcal{B}$ (for instance, B is an open set, or a compact set, etc...). The simplest relationships we may consider for structuring the set A are the following : $B \subset A$ (B is included into A), or $A \cap B \neq \emptyset$ (A hits B), and also all the relationships R we can obtain by applying the logical operations "and", "or", and "no" to the preceding ones. If we are able to say for any such relationship R if R is true or false, we may consider the structure of the medium A as perfectly defined.

This definition is formally correct. But practically, there are too many possible relationships R, and it is not possible to examine each of them individually. Some statistical treatment is required in order to condense this information. It is not really interesting to know if a given point x belongs or not to A, or if a given ball B hits or not the grains of our porous medium. All we need to know is the probability of such events, or the frequency of their occurence in our medium (these two points of view being equivalent if we assume hypotheses of stationarity and ergodicity).

Thus our medium A is now considered as a realization of a random set, and as such it is perfectly defined if we know for each possible relationship R the probability P(R) for R to be verified. This probabilistic point of view implies a change of scale. We are leaving a microscopic (or granulometric) level, on which the relationships R possess an individual meaning, in order to rise up to a macroscopic level where our medium appears as homogenized and simplified. On this macroscopic level, the probabilities P(R) acquire the meaning of precise physico-geometrical concepts (deterministic, and no longer probabilistic concepts). Let us give elementary examples, [2], [6] etc...

If the structuring figure B is reduced to a single point x, we obtain the probability $P(x \in A^c)$ for x to belong to the pores A^c . It is the <u>porosity</u> of our medium.

If we consider now two points x and x+h, we obtain the <u>covariance</u> function C(h) which already carries some significant structural information. It is well known in stereology that derivating C(h) in the neighbourhood of the origin leads to the <u>specific</u> area. On the other hand, by integrating C(h), we can obtain the <u>range</u> of our medium, i.e. the dimensions of hypergranulometric structures, if any. Finally, the covariance plays an important part in estimation problems, for it enables us to compute <u>estimation variances</u>.

Let us give a few more examples. With four points, it is possible to determine the <u>Euler-Poincaré characteristics</u> of our medium.

With a line segment, we obtain the well known linear granulometries, and, in connection with them, parameters like the stars which give the average dimensions of the pores as well as of the grains. More generally, with convex sets B (circles, hexagons and so on) we can determine pluridimensional granulometries (see below).

These probabilities P(R), and the physico-geometrical parameters connected with them, are easy to determine experimentally with the aid of the texture analyzer, [4], [5]. But the notion of random set we have just introduced in an heuristic manner is not always sufficient, even for practical applications, and it is necessary to give more precise mathematical statements. From a theoretical point of view, this will lead us to enlightening comparisons with integral geometry.

2 - RANDOM SETS AND INTEGRAL GEOMETRY

The mathematical notion of set is probably too rich for being directly applied to empirical realities. In the case of a porous medium, we have introduced above the set A of the points belonging to the grains. But, from a topological point of view, we may ask if A is to be considered as an open or a closed set, or as something more complicated. In other words, does a boundary point belong to the grains or to the pores ? From an experimental point of view, this question is clearly meaningless, because the notion of a point belonging to the boundary of A does not correspond to any physical reality. For an experimentalist knows no individual points, but only little spots with ill-defined boundaries. In first approximation, we may take this circumstance into account by considering only open sets as possible structuring figures B.

But, if B is an open set, B is included into A if and only if B is included into the interior $\mathring{A}$ of A, and B hits A if and only if B hits the closure $\bar{A}$ of A. Then the inclusion logics, founded on relationships of the type $B \subset A$ does not concern the set A itself, but only its interior, and in the same way, the intersection logics concerns only the closure $\bar{A}$ of A. Practically, these two complementary logics are always used simultaneously. But, for simplicity purposes, we shall restrict ourselves only to intersection logics in what follows. Consequently, instead of the too general notion of random set, we will consider only the concept of Random Closed Set (RCS), [1], [7] .

2-1 - Definition of a Random Closed Set (RCS).

Let us denote by $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{K}$ respectively the families of the closed, open and compact sets in the n-dimensional euclidean space E_n . If B is a subset of E_n (a structuring figure, for instance), we denote V_B the class of closed sets hitting B, and V^B its complementary :

$$V_B = \{F : F \in \mathcal{F} , F \cap B \neq \emptyset\} , \quad V^B = \{F : F \in \mathcal{F} , F \cap B \neq \emptyset\}$$

The intersection logics is founded on the V_B with $B \in \mathcal{G}$, for at the start, we are limiting ourselves to open structuring sets. In order to define the notion of RCS, we must at first introduce a σ-algebra on $\mathcal{F}$. We shall use the σ-algebra $\sigma(\mathcal{O})$ generated by the V_B , $B \in \mathcal{G}$. Following this definition, the events "B hits A", $B \in \mathcal{G}$ are measurable for $\sigma(\mathcal{O})$, and the same is true for any event we can obtain by applying any enumerable sequence of logical operations to the preceding ones. In order to obtain the precise mathematical definition of a RCS A, it remains to introduce a probability P on the measurable space $(\mathcal{F}, \sigma(\mathcal{O}))$. To any event $V \in \sigma(\mathcal{O})$ is then associated the possible relationship $A \in V$, and the probability $P(V) = P(A \in V)$ for this relationship to be true. Conversely, if we restrict ourselves to the intersection logics, the only possible relationships R are of this type, and thus are probabilisable.

It remains to verify that it is actually possible to construct probabilities on such a rich σ-algebra. It can be done by using certain properties of compacity, closely connected indeed with integral geometry. Our σ-algebra is rich enough to contain events like "A hits K", for K compact. Conversely, it turns out that the σ-algebra generated on $\mathcal{F}$ by these events V_K, $K \in \mathcal{K}$ is identical with $\sigma(\mathcal{O})$ itself. But the events V_G, $G \in \mathcal{G}$ and V^K, $K \in \mathcal{K}$ are also generating a topology for which the space $\mathcal{F}$ possesses good properties. In particular, $\mathcal{F}$ is a compact space. Thus, $\sigma(\mathcal{O})$ appears as the borelian tribe associated with a compact space, and consequently it is actually possible to build probabilities on this σ-algebra.

In the same way, the topology generated by the V_G, $G \in \mathcal{G}$ and the V^F, $F \in \mathcal{F}$ can be introduced on the space $\mathcal{K}$ of the compact sets. This topology is locally compact, and can be identified with the topology defined on $\mathcal{K}$ by the classical Hausdorff metrics. The associated borelian tribe leads to the notion of random compact set. But this notion is perhaps less interesting for the applications, because obviously a random compact set cannot be stationary ([7], [10], [11]).

2-2 - The Functional T(K).

It is well known that the probability associated with an ordinary random variable is entirely determined if the corresponding distribution function is given. There is a similar circumstance for the random closed sets. If A is a RCS and P the associated probability on $\sigma(\mathcal{O})$, let us denote by :

$$T(K) = P(V_K) = P(A \cap K \neq \emptyset)$$

the probability for A to hit a given compact $K \in \mathcal{K}$. We obtain a functional T on $\mathcal{K}$ associated with the probability P. Conversely, it can be shown that the probability P on $\sigma(\mathcal{O})$ is entirely determined if the functional T on $\mathcal{K}$ is given. In respect to the RCS A, T is thus similar to the distribution function of a Random Variable, [10] .

The functionals defined on the space $\mathcal{K}$ of the compact sets are of great importance in integral geometry, [3] . Thus it is interesting to find the necessary and sufficient conditions for a given functional T to be associated with a RCS. At first, it is required that T be null for the empty set, $T(\emptyset) = 0$, because the empty set does not hit any other set, and also $T \leq 1$. If K_n is a decreasing sequence of compact sets and $K = \bigcap K_n$ its intersection, it can be shown that the sequence V_{K_n} is also decreasing in $\mathfrak{F}$ and its intersection is V_K . Thus, the sequence $T(K_n)$ must converge toward $T(K)$ (sequential continuity). This property appears to be equivalent to upper semi-continuity of the functional T on $\mathcal{K}$ for the classical Hausdorff metrics. Finally, let us denote by $S_n\ (K_o;\ K_1, \ldots, K_n)$ the probability for A not to hit the compact K_o, but to hit the other compacts $K_1, \ldots, K_n$. These functions are obtained by the following recurrence formulae :

$$S_1\ (K_o\ ; K_1) = T\ (K_o \cup K_1) - T(K_o)$$

$$S_n(K_o; K_1, \ldots, K_n) = S_{n-1}(K_o; K_1, \ldots, K_{n-1}) - S_{n-1}(K_o \cup K_n; K_1, \ldots K_{n-1}$$

Clearly, these functions S_n must be ≥ 0 for any integer n and any compacts $K_o, K_1, \ldots K_n$.

These three conditions must be verified by the functional T. We can summarize the two most important of them (sequential continuity and positivity of the S_n) by saying that T is an alternating capacity of infinite order . It turns out that the converse is true, and we may state the following :

Theorem 1 (Choquet) : A functional T on $\mathcal{K}$ is associated with a random closed set A verifying $P(A \cap K \neq \emptyset) = T(K)$ for any compact K if and only if T is an alternating capacity of infinite order verifying $T(\emptyset) = 0$ and $T \leq 1$.

This theorem goes back to G. Choquet (1953) [1]. Its greatest interest for us is to determine the class of functionals T on $\mathcal{K}$ admitting a probabilistic interpretation, and to tie together integral geometry and the random sets theory. It turns out that many properties, classical in integral geometry, admit a probabilistic interpretation, and conversely. Let us give a few examples, [11] .

Let us consider at first the case where the functional T is strongly additive on the class of the convex compact sets [3], i.e. verifies :

$$T(K \cup K') + T(K \cap K') = T(K) + T(K')$$

if K, K' and their union $K \cup K'$ are compact and convex. If the conditions of the Choquet theorem are also verified by T, then the associated random closed set A is almost surely convex, and the converse is also true : thus, the strong additivity of the functional T and the almost sure convexity of the random set A can be identified.

2-3 - RCS indefinitely divisible for the Union.

A random closed set A is indefinitely divisible for the Union if for any integer n A is equivalent to the union $A_1 \cup \ldots \cup A_n$ of n independent and equivalent RCS $A_1, \ldots A_n$. Let us denote by

$$Q(K) = 1 - T(K) = P(A \cap K) = \emptyset)$$

the probability for A not to hit the compact K. Clearly, A is indefinitely divisible if and only if the conditions of the Choquet theorem are verified by the functionals

$$T_n = 1 - Q^{1/n}$$

In order to eliminate unessential complications, let us also assume that there is no fixed points (i.e. no points x such that $P(x \in A) = 1$). Then, we have the following theorem, [11] :

Theorem 2 : A functional T on $\mathcal{K}$ is associated with an indefinitely divisible RCS without fixed points if and only if there exists an alternating capacity of infinite order ψ verifying $\psi(\emptyset) = 0$ and $Q(K) = 1 - T(K) = \exp\{-\psi(K)\}$ for any compact K.

The limitation $T \leq 1$ is no longer required, and we obtain a larger class of functionals T. For instance, the Poisson flats of R.E. Miles [12], [13], [10] are the indefinitely divisible random cmosed sets associated with functionals ψ, the restriction of which on the convex compacts are identical (in the isotropic case) to the notorious Minkowski functionals (Quermassintegrale)

A more general example is given by the boolean schemes, [6], [10]. In order to obtain a boolean scheme, we assume that a Poisson process is given in the euclidean space. Independent primary RCS, equivalent up to a translation, are located at each Poisson point, and we consider the union of these primary sets. This union is a random (boolean) indefinitely divisible closed set, the functional ψ of which is given by :

$$\psi(K) = \theta \; E[\lambda(A' \oplus \check{K})] \qquad (K \in \mathcal{K})$$

θ denotes the density of the Poisson process, E the expectation, λ the Lebesgue measure, A' the primary random set located at the origin, $\oplus$ the Minkowski addition, and $\check{K}$ the symmetric of K in respect to the origin.

2-4 - The semi-Markovian property.

The semi-markovian property, [6], [10], leads to another interpretation for the strongly additive functionals of the integral geometry. Let us say that two compacts K and K' are separated by a compact C if any segment (x,x') joining a point $x \in K$ to a point $x' \in K'$ hits the compact C. Then, a random closed set A is said to be semi-markovian if, for any compacts K and K' separated by another compact C, the RCS $A \cap K$ and $A \cap K'$ are conditionnally independent for $C \cap A = \emptyset$.

It can be shown [10] that A is semi-markovian if and only if its functional $Q = 1 - T$ verifies the relationship :

$$Q(K \cup K' \cup C) \; Q(C) = Q(K \cup C) \; Q(K' \cup C)$$

if K and K' are separated by C. In particular, if the compacts K and K', not necessarily convex themselves, have a convex union, they are separated by their intersection $K \cap K'$, and thus the semi-markovian property implies :

$$Q(K \cup K') \; Q(K \cap K') = Q(K) \; Q(K')$$

In other words, the functional $\psi = - \ell n \, Q$ is strongly additive. The converse is also true if A is indefinitely divisible. In other words, [11], we have the

Theorem 3 : An indefinitely divisible random closed set A is semi-markovian if and only if the functional on $\mathcal{K}$ defined by : $\psi(K) = - \ell n \, P(A \cap K = \emptyset)$ is strongly additive on the class of the convex compact sets.

The corresponding class of functionals ψ (alternating capacities of infinite order, null in $K = \emptyset$ and strongly additive for the convex compacts) is probably the most interesting one for integral geometry. In the particular case where A is stationary and isotropic (i.e. ψ invariant under the translations and the rotations), a classical theorem of integral geometry [3] shows that $\psi(K)$, for K convex compact, is given by the formula :

$$\psi(K) = \sum_{i=0}^{n} \beta_i \, W_i(K)$$

where $W_i(K)$ denotes the Minkowski functional of index i, and β_i are non negative coefficients.

The boolean schemes with convex primary grains [6], [10], [11], give a general example of semi-markovian RCS. It can be shown that any indefinitely divisible semi-markovian RCS is limit, in a certain sense, of such boolean schemes. (But I do not know if there exist semi-markovian RCS which are not indefinitely divisible). Figure 1 shows a realization of a boolean scheme in which the primary random sets are circles with fixed radii. (Figure 1)

The preceding examples intended to stress the connections between integral geometry and random sets theory. Let us now make a little more detailed study of an important notion : the granulometry.

3 - THE NOTION OF GRANULOMETRY

Generally, the usual concepts of granulometry (or size distribution) are not very well defined from a mathematical point of view, and do not clearly correspond to precise geometrical properties. On the other hand, they are relevant only for materials constituted by distinguishable grains (connex components). But very often the grains are connected, and even if they are not, it would be also interesting to get some information on the size distribution of the pores. For instance, in a porous medium, an hydrodynamic property like permeability is clearly connected with the dimensions of the pores rather than with the granulometry of the grains themselves. Thus, a mathematical definition of a granulometry needs to satisfy the two following conditions [6] :

a/ It must correspond to precise geometrical properties

b/ and be relevant even in the case of a connected medium, and for the pores as well as for the grains themselves.

The linear granulometry (or statistics of linear intercepts) gives us a first example of a satisfying definition. In what follows, we will try to build a general notion by using an axiomatic method.

3-1 - The axioms of granulometries.

In view of finding the suitable axioms, let us at first analyze the usual practice. Sieves are given, the mesh sizes of which are characterized by a parameter $\lambda > 0$. By applying the sieve λ to (a material idealized by) a set A, we obtain an oversize which is a subset $\psi_\lambda(A)$ of A, and thus we have $\psi_\lambda(A) \subset A$. If B is another set including A, the B oversize for a given mesh λ is clearly greater than the A oversize, i.e. : $\psi_\lambda(A) \subset \psi_\lambda(B)$ if $A \subset B$. In the same way, if we compare two different meshes λ and μ with $\lambda \geq \mu$, the μ oversize is greater than the λ oversize, i.e. : $\psi_\lambda(A) \subset \psi_\mu(A)$ if $\lambda \geq \mu$. Finally, with $\lambda \geq \mu$, we get the λ oversize $\psi_\lambda(A)$ of A itself by applying the greatest mesh λ to the μ oversize. In the same way, we obtain again $\psi_\lambda(A)$ itself by applying the smallest mesh μ to the λ oversize $\psi_\lambda(A)$.

If we take these four properties of usual granulometries as axioms, we obtain the following definition, [8] :

Definition : If E is a space and $\mathcal{A}$ a family of subsets of E, a granulometry on $\mathcal{A}$ is a family ψ_λ , $\lambda > 0$, of mappings from $\mathcal{A}$ into itself verifying the following axioms :

1/ $\varphi_\lambda(A) \subset A$ for any $A \in \mathcal{A}$ and $\lambda > 0$

2/ A, $B \in \mathcal{A}$ and $A \subset B$ implies :

$$\psi_\lambda(A) \subset \psi_\lambda(B)$$

3/ $A \in \mathcal{A}$ and $\lambda \geq \mu > 0$ implies :

$$\psi_\lambda(A) \subset \psi_\mu(A)$$

4/ if λ and μ are positive numbers :

$$\psi_\lambda \circ \psi_\mu = \psi_\mu \circ \psi_\lambda = \psi_{\mathrm{Sup}(\lambda,\mu)}$$

Axiom 3/ is not really necessary, because it follows from axioms 1/ and 4/, and is written only for clarity. If we complete our definition by putting $\psi_0(A) = A$ for $\lambda = 0$, axiom 1/ is implied by axiom 4/ . Thus, a granulometry is actually characterized by the only axioms 2/ and 4/ .

We note that the oversize $\psi_\lambda(A)$ is a <u>set</u> (and not a number). In order to obtain the usual granulometric (size distribution) curve, we may take the measure (the volume, or the weight, etc...) of $\psi_\lambda(A)$, if the set A is bounded. In the probabilistic formulation, we shall rather take the probability for a given point x to belong to $\psi_\lambda(A)$. This geometrical definition of $\psi_\lambda(A)$ is required by axiom 4/, which expresses the most characteristic property of granulometries.

Let us now analyze this definition. For a given λ, axioms 1/, 2/ and axiom 4/ written with $\mu = \lambda$ ($\psi_\lambda \circ \psi_\lambda = \psi_\lambda$) characterize ψ_λ as an algebraïc opening. Like any algebraïc opening, ψ_λ is determined by the family $\mathcal{B}_\lambda$ of its invariant sets (i.e., sets B such that $\psi_\lambda(B) = B$). For $\psi_\lambda(A)$ is the union of the invariant sets included into A :

(1) $$\psi_\lambda(A) = \bigcup \{B : B \in \mathcal{B}_\lambda \, , \; B \subset A\}$$

It can then easily be shown that the axioms 3/ and 4/ are also verified if and only if the families $\mathcal{B}_\lambda$ are decreasing with λ :

$$\mu \geq \lambda \Rightarrow \mathcal{B}_\mu \subset \mathcal{B}_\lambda$$

Thus formula (1) gives the general form of a granulometry.

<u>3-2 - Euclidean Granulometries</u>.

The preceding general definition is valid in any space E. if E is the n-dimensional space, we may encounter additional conditions. Usual granulometries are compatible with the translations : if A_h denotes the set A translated by a vector h, the oversize of A_h is the translated set $(\psi_\lambda(A))_h$

5/ $$\psi_\lambda(A_h) = (\psi_\lambda(A))_h$$

/

On the other hand, the usual sieves are homothetical. If we take as parameter λ the homothetic ratio between the mesh λ and a reference mesh (corresponding then to $\lambda = 1$), the oversize for the sieve λ of the homothetic λA of a set A is the homothetic $\lambda\, \psi_1(A)$ of the oversize $\psi_1(A)$ for the reference sieve.

$$6/ \qquad \psi_\lambda(\lambda A) = \lambda\, \psi_1(A)$$

We shall say that a granulometry ψ_λ is <u>euclidean</u> if it satisfies also the two additional axioms 5/ and 6/.

Axiom 5/ is verified if and only if the family $\mathcal{B}_\lambda$ is closed under translations. Axiom 6/ is verified if and only if $\mathcal{B}_\lambda = \lambda\, \mathcal{B}_1$ (i.e. $B \in \mathcal{B}_\lambda \Leftrightarrow \frac{1}{\lambda}\, B \in \mathcal{B}_1$). This family $\mathcal{B}_1$ is required to be closed under translation and homotheties with ratios ≥ 1, and these conditions are sufficient.

In other words, ψ_λ is an euclidean granulometry if and only if there exists <u>a family $\mathcal{B}$, closed for translations and homotheties with ratios ≥ 1</u>, such that :

$$(2) \qquad \psi_\lambda(A) = \bigcup \{\lambda B,\ B \in \mathcal{B},\ \lambda B \subset A\}$$

This is the general formula for euclidean granulometry. But it is possible to simplify it .

Let us at first introduce the notion of opening a set A following a set B, which will be denoted A_B . The definition of A_B is :

$$A_B = (A \ominus \check{B}) \oplus B$$

where $\oplus$ denotes the Minkowski addition, $\ominus$ the dual operation or Minkowski's subtraction, and B the symmetric of B. It can be shown that a point x belongs to A_B if and only if there exists a translated B_h of B including x and included into A:

$$x \in A_B \Leftrightarrow \exists\, h \in \mathbb{R}^n : x \in B_h \subset A$$

In other words, A_B is the domain covered by the translated sets B_h included into A. The mapping $A \rightarrow A_B$ is clearly an algebraic opening. It is the reason why A_B is said to be <u>the opening of A following B</u>. [6], [7] .

Let us now return to euclidean granulometry, and denote by $\mathcal{B}_0$ a subfamily of $\mathcal{B}$ such that the closure of $\mathcal{B}_0$ by unions, translations and homotheties with ratios ≥ 1 is identical to $\mathcal{B}$. Thus, the euclidean granulometry associated to $\mathcal{B}$ is given by the following formula [8] :

$$(3) \qquad \psi_\lambda(A) = \bigcup_{B \in \mathcal{B}_0} \bigcup_{\mu \geq \lambda} A_{\mu B}$$

Conversely, formula (3) gives the general form of euclidean granulometries.

In practice, $\mathcal{B}_0$ will always be a family of compact sets. For, if B is compact, A_B is closed when A is closed and open when A is open. If $\mathcal{B}_0$ verifies convenient topological properties, and for instance if $\mathcal{B}_0$ is finite, $\psi_\lambda(A)$ itself is closed if A is closed.

<u>3-3 - Granulometry of A following the set B.</u>

In practice, the most interesting case occurs if we choose a unique compact B for the family $\mathcal{B}_0$ appearing in formula (3). The corresponding euclidean granulometry is given by :

$$(4) \qquad \psi_\lambda(A) = \bigcup_{\mu \geq \lambda} A_{\mu B}$$

If the compact B is also <u>convex</u>, we get a very simple result. In this case, $A_{\lambda B}$ is a decreasing function of λ :

$$\lambda \geq \mu \Rightarrow A_{\lambda B} \subset A_{\mu B}$$

and we simply get :

(5) $$\psi_\lambda (A) = A_{\lambda B}$$

Conversely, it can be shown that the mapping $A \to A_{\lambda B}$ is a granulometry if and only if the compact B is convex. Thus, the remarkable simplicity of the granulometry (5) is closely connected with the convexity of the structuring element B.

The mapping $A \to A_{\lambda B}$ from $\mathfrak{F}$ into itself is upper semi-continuous, thus measurable for $\sigma(\mathcal{O})$, and thus the probability of the event $x \in A_{\lambda B}$ does exist. If we put :

$$1 - F_x(\lambda) = P(x \in A_{\lambda B})$$

we obtain the granulometric curve associated with the random closed set A and the structuring element B. This curve represents the size of the set A evaluated at the point x with the aid of the standard B. In the stationary case (the most interesting in practice), this curve does not depend on the point x, and expresses an intrinsic property of an homogeneous medium (i.e. its size distribution in respect to the standard B).

The function $F_x(\lambda)$ is also the distribution function associated to the random variable Λ (x) defined as follows :

$$\Lambda(x) = \text{Sup } \{\lambda : x \in A_{\lambda B}\}$$

This random variable represents the size of the greatest homothetic λB included into A and including the point x. Thus , the size distribution $F_x(\lambda)$ admits the following simple probabilistic interpretation :

$$1 - F_x(\lambda) = P(\Lambda(x) \geq \lambda)$$

If the structuring figure B is a line segment, we obtain the linear granulometries mentioned above. These linear granulometries are easily measured, but give only unidimensional information. With circles or balls, we obtain pluridimensional information. However, in practice hexagons are often used instead of circles, and the corresponding hexagonal granulometries are easily measured with the texture analyzer [5], [6].

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